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and phases in mechanics

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## ***Abstract***

Various holonomy phenomena are shown to be instances of the reconstruction procedure for mechanical systems with symmetry. We systematically exploit this point of view for fixed systems (for example with controls on the internal, or reduced, variables) and for slowly moving systems in an adiabatic context. For the latter, we obtain the phases as the holonomy for a connection which synthesizes the Cartan connection for moving mechanical systems with the Hannay-Berry connection for integrable systems. This synthesis allows one to treat in a natural way examples like the ball in the slowly rotating hoop and also non-integrable mechanical systems.

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## Introduction

This paper is concerned with the interpretation of the Hannay-Berry phase for classical mechanical systems as the holonomy of a connection on a bundle associated with the given problem. The techniques apply to the quantum case in the spirit of Aharonov and Anandan [1987], Anandan [1988] and Simon [1983] using the well known fact that quantum mechanics can be regarded as an instance of classical mechanics (see for instance Abraham and Marsden [1978]). In carrying this out there are a number of interesting new issues beyond that found in Hannay [1985] and Berry [1984], [1985] that arise. Already this is evident for the example of the ball in the hoop discussed in Berry [1985]; some remarks on this example are discussed in §1 below. For slowly varying integrable systems and for some aspects of the nonintegrable case, progress was made already by Golin, Knauf, and Marmi [1989] and Montgomery [1988]. The situation for the integrable case has been generalized to the context of families of Lagrangian manifolds by Weinstein [1989a,b]. However, these do not satisfactorily cover even the ball in the hoop example. For this and other examples, there is need for a development of the formulation, and it is the purpose of this paper to give one, following the line of investigation initiated by these papers. One of the crucial new ingredients in the present paper is the introduction of a connection that is associated to the movement of a classical system that we term the *Cartan connection*. It is related to the theory of classical spacetimes that was developed by Cartan [1923] (see for example, Marsden and Hughes [1983] for an account). Another ingredient is the systematic use of symmetry and reduction, which are the key concepts needed to generalize to the nonintegrable case. In fact it is through the reconstruction process that the holonomy enters.

The paper begins in §1 with some simple examples. The purpose is to give an idea of the Cartan connection. The first example is the ball in the hoop. The second example is the problem of two coupled rigid bodies to illustrate some of the ideas involved in reconstruction (here there are no slowly varying parameters, but there is still holonomy). We also give the Aharonov-Anandan formula for quantum mechanics (given in detail in §4) and a resume of slowly varying integrable systems from Golin, Knauf, and Marmi [1989] and Montgomery [1988]. Finally, we give the example of reconstructing the motion of a freely spinning rigid body.

§§2 and 3 deal with the general theory of reconstruction. Given a phase space  $P$  and a symmetry group  $G$ , we show how to reconstruct the dynamics on  $P$  from dynamics on the reduced spaces. If  $J : G \rightarrow \mathfrak{g}^*$  is an equivariant momentum map for the  $G$ -action, the reduced space is  $P_\mu = J^{-1}(\mu)/G_\mu$ , where  $G_\mu$  is the coadjoint isotropy at  $\mu$ . This reconstruction is done using a choice of connection on the principal  $G_\mu$ -bundle  $J^{-1}(\mu) \rightarrow P_\mu$  (assuming the action is free). In case  $P$  is a cotangent bundle, there is a family of natural choices of connections

depending on a choice of metric on the configuration space and on a choice of transverse cross section to the  $G_\mu$ -orbit. We shall later refer to this one as the *mechanical connection*. Another one is built out of the canonical one-form and applies when  $G_\mu$  is abelian. The mechanical connection was defined by Guichardet [1984] and is closely related to connections defined by Smale [1970] and Kummer [1981]. For the case of cotangent bundles of semisimple Lie groups, the first includes the second as a special case.

We treat both the Lagrangian and Hamiltonian cases, since in the former the procedure is considerably more concrete because the Euler-Lagrange equations are of second order. In these sections there are no slowly varying parameters, but there are connections and holonomy. The connections are combined with the Hannay-Berry construction in §14.

§4 gives, for the convenience of the reader, background material on Ehresmann connections, curvature and holonomy that is needed for the paper. It is illustrated with the Aharonov-Anandan formula and other examples of mechanical systems—such as the top in a gravitational field and coupled planar rigid bodies in §5. In §§6, 7 and 8 we present the basic defining properties and the existence and uniqueness of the Hannay-Berry connection. A crucial aspect of the construction is to take a given connection and average it relative to the action of a group  $G$ . This action also defines a parametrized momentum map  $I$ , which plays an important role. We generalize the theory developed in Montgomery [1988], and Golin, Knauf, and Marmi [1989] for trivial bundles with symplectic fibers and the standard connection to nontrivial fiber bundles whose fibers are Poisson manifolds and with a connection compatible with this structure. It is important to allow a nontrivial connection at this stage, even if the bundle is trivial, in order to deal with moving systems, like the ball in the hoop. For moving systems, the nontrivial connection used is the *Cartan connection* described in §11. §9 gives another way to look at the Hannay-Berry connection by utilizing the momentum map for the group  $G$ . §10 studies the important case of slowly moving integrable systems. This is the case that motivated the development in Montgomery [1988], Golin, Knauf, and Marmi [1989], and Weinstein [1989a,b]. We generalize this to our context.

§12 presents a general construction for inducing connections on a tower of two bundles:  $E \rightarrow F \rightarrow M$  with a given a connection on  $E \rightarrow M$  and a family of fiberwise connections on  $E \rightarrow F$ . This is applied in §13 with  $E = I^{-1}(\mu)$ ,  $F = I^{-1}(\mu)/G_\mu$ , and  $M$  the parameter space. This is a parametrized version of the bundle of reduced spaces. This construction allows us to glue together the Hannay-Berry connection and the connection on the bundle  $J^{-1}(\mu) \rightarrow P_\mu$  used in §§2 and 3 to obtain a connection on  $I^{-1}(\mu) \rightarrow I^{-1}(\mu)/G_\mu$ . The holonomy of this synthesized connection gives the desired phase changes in many of the equations.

In this paper, there are three lines of investigation one can focus on if desired. We regard §4 on Ehresmann connections as necessary background for all three. The three lines are:

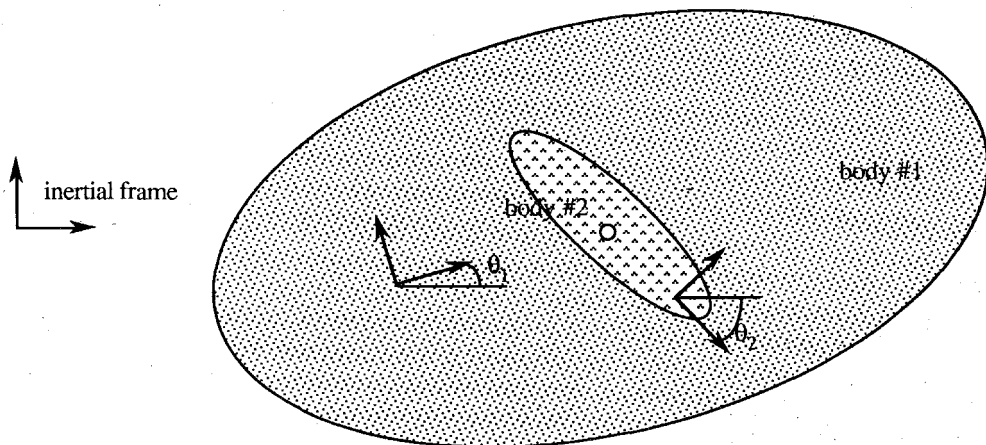
- 1 Reconstruction ideas: §1C, D, F, §2, 3, 5, 13
- 2 Adiabatic phases and moving systems: §1A, B, E, 6, 7, 8, 9, 10, 11, 12
- 3 Synthesis and future directions: §13, 14.

## §1 Some Examples

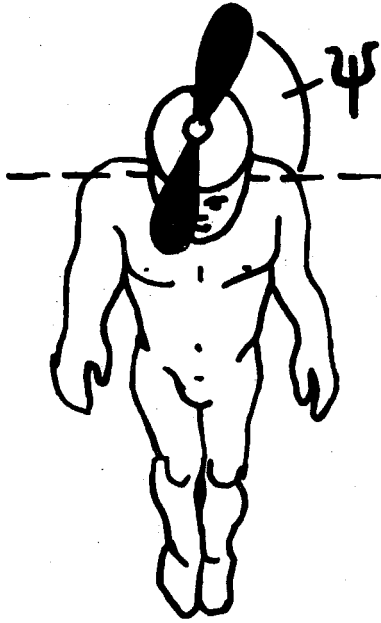
In this section we present some elementary examples exhibiting the general geometric features that will be discussed in the body of the paper. They focus on the ideas of reconstruction of dynamics and the "phases" obtained when reconstruction is performed on a closed loop. In this case, we shall distinguish between a geometric and a dynamic phase. Such phenomena naturally occur in Hamiltonian systems depending on a parameter, for example, in moving systems or in integrable systems depending on a "slow" parameter. The reader will find additional examples in §5. In particular, the rotating top in a gravitational field (the heavy top) and the dynamics of a system of planar coupled rigid bodies are treated there. The formula for the phase of the system of coupled planar rigid bodies is first computed by hand in §5E, so this can be read as part of the present section if desired.

Before beginning any serious examples, we will give an elementary example—Elroy's beanie—which still illustrates many of the interesting features of more complicated examples. In general, the theory and examples in this work can be divided into two types—those involving adiabatic phenomena and those that are "pure mechanical" or "pure reconstruction". Our first main example on moving systems in §1A is of the adiabatic type, while Elroy's beanie is purely mechanical.

**Example—Elroy's Beanie** Consider two planar rigid bodies joined by a pin joint *at their center of masses*. Let  $I_1$  and  $I_2$  be their moments of inertia, and  $\theta_1$  and  $\theta_2$  be the angle they make with a fixed inertial direction, as in the figure.



Elroy's Beanie



-PM

### Elroy and his beanie

Conservation of angular momentum states that  $I_1 \dot{\theta}_1 + I_2 \dot{\theta}_2 = \mu = \text{constant in time}$ , where the overdot means time derivative. The *shape space* of a system is the space whose points give the shape of the system. In this case, shape space is the circle  $S^1$  parametrized by the *hinge angle*  $\psi = \theta_2 - \theta_1$ . We parametrize the configuration space of the system not by  $\theta_1$  and  $\theta_2$  but by  $\theta = \theta_1$  and  $\psi$ . Conservation of angular momentum reads

$$I_1 \dot{\theta} + I_2 (\dot{\theta} + \dot{\psi}) = \mu; \quad \text{that is,} \quad d\theta + \frac{I_2}{I_1 + I_2} d\psi = \frac{\mu}{I_1 + I_2} dt. \quad (1)$$

The left hand side of (1) is the *mechanical connection* discussed in detail in §2.4. Suppose that the beanie (body #2) goes through one full revolution so that  $\psi$  increases from 0 to  $2\pi$ . Suppose, moreover, that the total angular momentum is zero:  $\mu = 0$ . From (1) we see that the entire configuration undergoes a net rotation of



$$\Delta\theta = -\frac{I_2}{I_1 + I_2} \int_0^{2\pi} d\psi = -\left(\frac{I_2}{I_1 + I_2}\right) 2\pi. \quad (2)$$

This is the amount by which Elroy rotates, each time his beanie goes around once.

Notice that the result (2) is independent of the detailed dynamics and only depends on the fact that angular momentum is conserved and the beanie goes around once. In particular, we get the same answer even if there is a "hinge potential" hindering the motion or if there is a control present in the joint. Also note that if Elroy wants to rotate by  $-2\pi k \frac{I_2}{I_1 + I_2}$  radians, where  $k$  is an integer, all he needs to do is spin his beanie around  $k$  times, then reach up and stop it. By conservation of angular momentum, he will stay in that orientation after stopping the beanie.

Here is a geometric interpretation of this calculation. The connection we used is  $A_{\text{mech}} = d\theta + \frac{I_2}{I_1 + I_2} d\psi$ . This is a flat connection for the trivial principal  $S^1$ -bundle  $\pi: S^1 \times S^1 \rightarrow S^1$  given by  $\pi(\theta, \psi) = \psi$ . Formula (2) is the holonomy of this connection, when we traverse the base circle,  $0 \leq \psi \leq 2\pi$ . (We note that this is the same connection that appears in the Aharonov-Bohm effect.)

### §1A Moving systems

Begin with a reference configuration  $Q$  and a Riemannian manifold  $S$ . Let  $M$  be a space of embeddings of  $Q$  into  $S$  and let  $m_t$  be a curve in  $M$ . If a particle in  $Q$  is following a curve  $q(t)$ , and if we imagine the configuration space  $Q$  moving by the motion  $m_t$ , then the path of the particle in  $S$  is given by  $m_t(q(t))$ . Thus, its velocity in  $S$  is given by the time derivative:

$$T_{q(t)} m_t \cdot \dot{q}(t) + Z_t(m_t(q(t))) \quad (1)$$

where  $Z_t$ , defined by  $Z_t(m_t(q)) = \frac{d}{dt} m_t(q)$ , is the time dependent vector field (on  $S$  with domain  $m_t(Q)$ ) generated by the motion  $m_t$  and  $T_{q(t)} m_t \cdot w$  is the derivative (tangent) of the map  $m_t$  at the point  $q(t)$  in the direction  $w$ . To simplify the notation, we write

$$m_t = T_{q(t)} m_t \quad \text{and} \quad q(t) = m_t(q(t)).$$

Consider a Lagrangian on  $TQ$  of the form kinetic minus potential energy. Using (1), we thus choose

$$L_{m_t}(q, v) = \frac{1}{2} \|m_t \cdot v + Z_t(q(t))\|^2 - V(q) - U(q(t)) \quad (2)$$

where  $V$  is a given potential on  $Q$  and  $U$  is a given potential on  $S$ .

Put on  $Q$  the (possibly time dependent) metric induced by the mapping  $m_t$ . In other words, we choose the metric on  $Q$  that makes  $m_t$  into an isometry for each  $t$ . In many examples of mechanical systems, such as the ball in the hoop given below,  $m_t$  is already a restriction of an isometry to a submanifold of  $S$ , so the metric on  $Q$  in this case is in fact time independent. Now we take the Legendre transform of (2), to get a Hamiltonian system on  $T^*Q$ . Recall (see, for example, Abraham and Marsden [1978] or Arnold [1978]), that the Legendre transformation is given by  $p = \frac{\partial L}{\partial v}$ . Taking the derivative of (2) with respect to  $v$  in the direction of  $w$  gives:

$$p \cdot w = \langle m_t \cdot v + Z_t(q(t)), m_t \cdot w \rangle_{q(t)} = \langle m_t \cdot v + Z_t(q(t))^T, m_t \cdot w \rangle_{q(t)} \quad (3a)$$

where  $p \cdot w$  means the natural pairing between the covector  $p \in T_{q(t)}^*Q$  and the vector  $w \in T_{q(t)}Q$ .  $\langle, \rangle_{q(t)}$  denotes the metric inner product on the space  $S$  at the point  $q(t)$  and  $T$  denotes the tangential projection to the space  $m_t(Q)$  at the point  $q(t)$ . Recalling that the metric on  $Q$ , denoted  $\langle, \rangle_{q(t)}$  is obtained by declaring  $m_t$  to be an isometry, (3a) gives

$$p \cdot w = \langle v + m_t^{-1} Z_t(q(t))^T, w \rangle_{q(t)} \quad \text{i.e.,} \quad p = (v + m_t^{-1} Z_t(q(t))^T)^b \quad (3b)$$

where  $^b$  denotes the index lowering operation at  $q(t)$  using the metric on  $Q$ . The (in general time dependent) Hamiltonian is given by the prescription  $H = p \cdot v - L$ , which in this case becomes

$$\begin{aligned} H_{m_t}(q, p) &= \frac{1}{2} \|p\|^2 - \mathcal{P}(Z_t) - \frac{1}{2} \|Z_t^\perp\|^2 + V(q) + U(q(t)) \\ &= H_0(q, p) - \mathcal{P}(Z_t) - \frac{1}{2} \|Z_t^\perp\|^2 + U(q(t)), \end{aligned} \quad (4)$$

where  $H_0(q, p) = \frac{1}{2} \|p\|^2 + V(q)$ , the time dependent vector field  $Z_t \in \mathfrak{X}(Q)$  is defined by  $Z_t(q) = m_t^{-1}[Z_t(m_t(q))]^T$ , the momentum function  $\mathcal{P}(Y)$  is defined by  $\mathcal{P}(Y)(q, p) = p \cdot Y(q)$  for  $Y \in \mathfrak{X}(Q)$ , and where  $Z_t^\perp$  denotes the orthogonal projection of  $Z_t$  to  $m_t(Q)$ . Even though the Lagrangian and Hamiltonian are time dependent, we recall that the Euler-Lagrange equations for  $L_{m_t}$  are equivalent to Hamilton's equations for  $H_{m_t}$ . These give the correct equations of motion for this moving system. (An interesting example of this is fluid flow on the rotating earth, where it is important to consider the fluid with the motion of the earth superposed, rather than the motion relative to an observer.)

Let  $G$  be a Lie group that acts on  $Q$ . (For the ball in the hoop, this will be the dynamics of  $H_0$  itself). We assume for the general theory that  $H_0$  is  $G$ -invariant. Assuming the "averaging principle" (cf. Arnold [1978], for example) we replace  $H_{m_t}$  by its  $G$ -average,

$$\langle H_{m_t} \rangle(q, p) = \frac{1}{2} \|p\|^2 - \langle \mathcal{P}(Z_t) \rangle - \frac{1}{2} \langle \|Z_t^\perp\|^2 \rangle + V(q) + \langle U(q(t)) \rangle. \quad (5)$$

where  $\langle \cdot \rangle$  denotes the  $G$ -average. This principle is hard to justify in general and is probably only justified for torus actions for integrable systems. We will only use it in this case in examples, so we proceed with the form (5). Furthermore, we shall discard the term  $\frac{1}{2} \langle \|Z_t^\perp\|^2 \rangle$ ; we assume it is small compared to the rest of the terms. Thus, define

$$\mathcal{H}(q, p, t) = \frac{1}{2} \|p\|^2 - \langle \mathcal{P}(Z_t) \rangle + V(q) + \langle U(q(t)) \rangle = H_0(q, p) - \langle \mathcal{P}(Z_t) \rangle + \langle U(q(t)) \rangle. \quad (6)$$

The dynamics of  $\mathcal{H}$  on the extended space  $T^*Q \times M$  is given by the vector field

$$(X_{\mathcal{H}}, Z_t) = \left( X_{H_0} - X_{\langle \mathcal{P}(Z_t) \rangle} + X_{\langle U \circ m_t \rangle}, Z_t \right). \quad (7)$$

The vector field

$$\text{hor}(Z_t) = \left( -X_{\langle \mathcal{P}(Z_t) \rangle}, Z_t \right) \quad (8)$$

has a natural interpretation as the horizontal lift of  $Z_t$  relative to a connection, which we shall call the *Hannay-Berry connection induced by the Cartan connection*; see §11 and §12, especially Theorem 11.3. The holonomy of this connection is interpreted as the Hannay-Berry phase of a slowly moving constrained system. Let us give a few more details for the case of the ball in the rotating hoop.

### §1B The ball in the rotating hoop

In the following example, we follow some ideas of J. Anandan.

Consider Figure 1B-1 which shows a hoop (not necessarily circular) on which a bead slides without friction. As the bead is sliding, the hoop is slowly rotated in its plane through an angle  $\theta(t)$  and angular velocity  $\omega(t) = \dot{\theta}(t) \mathbf{k}$ . Let  $s$  denote the arc length along the hoop, measured from a reference point on the hoop and let  $\mathbf{q}(s)$  be the vector from the origin to the corresponding point on the hoop; thus the shape of the hoop is determined by this function  $\mathbf{q}(s)$ . The unit tangent vector is  $\mathbf{q}'(s)$  and the position of the reference point  $\mathbf{q}(s(t))$  relative to an inertial frame in space is  $R_{\theta(t)}\mathbf{q}(s(t))$ , where  $R_{\theta}$  is the rotation in the plane of the hoop through an angle  $\theta$ .

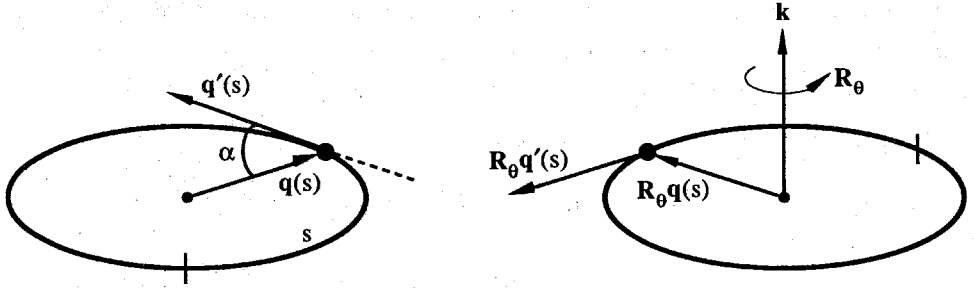


Figure 1B-1

The configuration space is diffeomorphic to the circle  $Q = S^1$  with length  $L$  the length of the hoop. The Lagrangian  $L(s, \dot{s}, t)$  is simply the kinetic energy of the particle; *i.e.*, since

$$\frac{d}{dt} R_{\theta(t)} q(s(t)) = R_{\theta(t)} q'(s(t)) \dot{s}(t) + R_{\theta(t)} [\omega(t) \times q(s(t))],$$

we set

$$L(s, \dot{s}, t) = \frac{1}{2} m \| q'(s) \dot{s} + \omega \times q(s) \|^2. \quad (1)$$

The Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{s}} = \frac{\partial L}{\partial s}$$

become

$$\frac{d}{dt} m[\dot{s} + q' \cdot (\omega \times q)] = m[\dot{s} q'' \cdot (\omega \times q) + \dot{s} q' \cdot (\omega \times q') + (\omega \times q) \cdot (\omega \times q')]$$

since  $\|q'\|^2 = 1$ . Therefore

$$\ddot{s} + q'' \cdot (\omega \times q) \dot{s} + q' \cdot (\dot{\omega} \times q) = \dot{s} q'' \cdot (\omega \times q) + (\omega \times q) \cdot (\omega \times q')$$

*i.e.*,

$$\ddot{s} - (\omega \times q) \cdot (\omega \times q') + q' \cdot (\dot{\omega} \times q) = 0. \quad (2)$$

The second and third terms in (2) are the centrifugal and Euler forces respectively. We rewrite (2) as

$$\ddot{s} = \omega^2 q \cdot q' - \dot{\omega} q \sin \alpha \quad (3)$$

where  $\alpha$  is as in Figure 1-1 and  $q = \|q\|$ . From (3), Taylor's formula with remainder gives

$$s(t) = s_0 + \dot{s}_0 t + \int_0^t (t - t') \left\{ \omega(t')^2 q \cdot q'(s(t')) - \dot{\omega}(t') q(s(t')) \sin \alpha(s(t')) \right\} dt'. \quad (4)$$

Now  $\omega$  and  $\dot{\omega}$  are assumed small with respect to the particle's velocity, so by the averaging theorem (see, e.g. Hale [1969]), the  $s$ -dependent quantities in (4) can be replaced by their averages around the hoop:

$$s(T) \approx s_0 + \dot{s}_0 T + \int_0^T (T-t') \left\{ \omega(t')^2 \frac{1}{L} \int_0^L \mathbf{q} \cdot \mathbf{q}' ds - \dot{\omega}(t') \frac{1}{L} \int_0^L q(s) \sin \alpha ds \right\} dt'. \quad (5)$$

**Aside** The essence of the averaging can be seen as follows. Suppose  $g(t)$  is a rapidly varying function and  $f(t)$  is slowly varying on an interval  $[a, b]$ . Over one period of  $g$ , say  $[\alpha, \beta]$ , we have

$$\int_{\alpha}^{\beta} f(t)g(t)dt \approx \int_{\alpha}^{\beta} f(t)\bar{g} dt \quad (6)$$

where  $\bar{g} = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} g(t)dt$  is the average of  $g$ . The error in (6) is

$$\int_{\alpha}^{\beta} f(t)(g(t) - \bar{g})dt$$

which is less than  $(\beta - \alpha) \times (\text{variation of } f) \times \text{constant} \leq \text{constant } |f'|(\beta - \alpha)^2$ . If this is added up over  $[a, b]$  one still gets something small as the period of  $g \rightarrow 0$ . ♦

The first integral in (5) over  $s$  vanishes and the second is  $2A$  where  $A$  is the area enclosed by the hoop. Now integrate by parts:

$$\int_0^T (T-t') \dot{\omega}(t') dt' = -T\omega(0) + \int_0^T \omega(t') dt' = -T\omega(0) + 2\pi, \quad (7)$$

assuming the hoop makes one complete revolution in time  $T$ . Substituting (7) in (5) gives

$$s(T) = s_0 + \dot{s}_0 T + \frac{2A}{L} \omega_0 T - \frac{4\pi A}{L}. \quad (8)$$

The initial velocity of the ball *relative to the hoop* is  $\dot{s}_0$ , while that *relative to the inertial frame* is (see (1)),

$$v_0 = \mathbf{q}'(0) \cdot [\mathbf{q}'(0) \dot{s}_0 + \omega_0 \times \mathbf{q}(0)] = \dot{s}_0 + \omega_0 q(s_0) \sin \alpha(s_0). \quad (9)$$

Now average (8) and (9) over the *initial conditions* to get

$$\langle s(T) - s_0 - v_0 T \rangle = -\frac{4\pi A}{L} \quad (10)$$

which means that *on average*, the shift in position is by  $\frac{4\pi A}{L}$  between the rotated and nonrotated hoop. Note that if  $\omega_0 = 0$  (the situation assumed by Berry [1985]) then averaging over initial conditions is not necessary. This process of averaging over the initial conditions that we naturally encounter in this example is related to the recent work of Golin and Marmi [1989] on experimental procedures to measure the phase shift.

This extra length  $\frac{4\pi A}{L}$  is sometimes called the *Hannay-Berry phase*. Expressed in angular measure, it is  $\frac{8\pi^2 A}{L^2}$ . In §11B we show, using the Cartan connection, how to realize this answer as the holonomy of the associated Hannay-Berry connection.

### §1C Coupled planar pendula

We return now to an example similar to Elroy's beanie, with which we began. Consider two coupled pendula in the plane moving under the influence of a potential depending on the hinge angle between them. Let  $r_1, r_2$  be the distances from the joint to their centers of mass and let  $\theta_1$  and  $\theta_2$  be the angles formed by the straight lines through the joint and their centers of mass relative to an inertial coordinate system fixed in space, as in Figure 1C-1. The Lagrangian of this system is

$$L = \frac{1}{2} m_1 r_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 r_2^2 \dot{\theta}_2^2 - V(\theta_1 - \theta_2)$$

and is therefore of the form kinetic minus potential energy, where the kinetic energy is given by the metric on  $\mathbb{R}^2$

$$ds^2 = m_1 r_1^2 d\theta_1^2 + m_2 r_2^2 d\theta_2^2 .$$

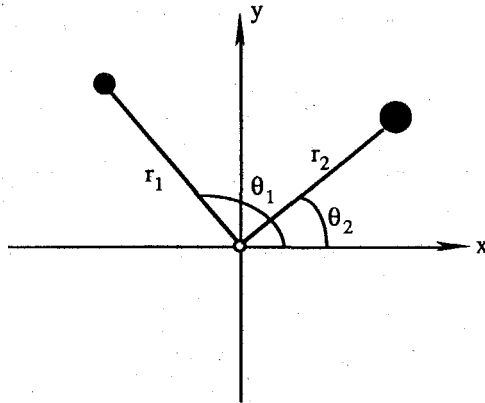


Figure 1C-1

To illustrate the ideas, we look at the special case  $m_1 = r_1 = 1$  so that

$$L = \frac{1}{2} (\dot{\theta}_1^2 + \dot{\theta}_2^2) - V(\theta_1 - \theta_2)$$

and

$$ds^2 = d\theta_1^2 + d\theta_2^2 = \frac{1}{2} d(\theta_1 - \theta_2)^2 + \frac{1}{2} d(\theta_1 + \theta_2)^2.$$

The group  $S^1$  acts on configuration space  $T^2 = \{(\theta_1, \theta_2)\}$  by  $\theta \cdot (\theta_1, \theta_2) = (\theta + \theta_1, \theta + \theta_2)$  so  $L$  is invariant under this action. Letting  $\phi = (\theta_1 + \theta_2)/\sqrt{2}$  and  $\psi = (\theta_1 - \theta_2)/\sqrt{2}$ , we see that  $\theta \cdot (\phi, \psi) = (\phi + \theta, \psi)$  and hence that the induced momentum map for the lifted action  $J : T^*T^2 \rightarrow \mathbb{R}$  is given by  $J(\phi, \psi, p_\phi, p_\psi) = p_\phi$ . Therefore the reduced space  $J^{-1}(\mu)/S^1$  is diffeomorphic to  $T^*S^1 = \{(\psi, p_\psi)\}$  with the canonical symplectic structure. The Hamiltonian on  $T^*T^2$  is

$$H(\phi, \psi, p_\phi, p_\psi) = \frac{1}{2} (p_\phi^2 + p_\psi^2) + V(\sqrt{2} \psi)$$

and the reduced Hamiltonian is

$$H_\mu(\psi, p_\psi) = \frac{1}{2} p_\psi^2 + V(\sqrt{2} \psi).$$

The equations of motion for  $H$  are

$$\dot{\phi} = p_\phi, \quad \dot{p}_\phi = 0 \tag{1}$$

$$\dot{\psi} = p_\psi, \quad \dot{p}_\psi = -\sqrt{2} V'(\sqrt{2} \psi). \tag{2}$$

Equations (2) are Hamilton's equations for  $H_\mu$  on the reduced space.

Assume that we have solved (2) with initial conditions  $(\psi_0, p_{\psi_0})$  and are given the initial conditions  $(\phi_0, \psi_0, p_{\phi_0} = \mu, p_{\psi_0})$  of a solution for the system (1), (2). To find the solution for  $(\phi(t), \psi(t), p_\phi(t), p_\psi(t))$  of (1), (2) we proceed in two steps:

**Step 1** Consider the curve  $d(t) = (\phi_0, \psi(t), \mu, p_\psi(t))$ .

**Step 2** Solve the equation  $\theta'(t) = \mu, \theta(0) = 0$  yielding  $\theta(t) = \mu t$ .

Then the solution to (1), (2) is given by  $c(t) = \theta \cdot d(t) = (\phi_0 + \mu t, \psi(t), \mu, p_\psi(t))$ .

This method is quite general and applies to all Hamiltonian systems. We will discuss it in §2 and §3. To get a feeling of what is happening, we make some remarks. The principal  $S^1$ -bundle  $J^{-1}(\mu) = \{(\phi, \psi, \mu, p_\psi)\} \rightarrow T^*S^1, (\phi, \psi, \mu, p_\psi) \mapsto (\psi, p_\psi)$  has a connection whose horizontal space at any point is generated by the vector fields  $\left\{ \frac{\partial}{\partial \psi}, \frac{\partial}{\partial p_\psi} \right\}$ . Then the curve  $d(t)$  in

**Step 1** is simply the horizontal lift of the integral curve of the reduced system  $(\psi(t), p_\psi(t))$

through  $(\varphi_0, \psi_0, \mu, p_{\psi_0})$ . Note that this connection on  $J^{-1}(\mu) \rightarrow T^*S^1$  is the pull-back of the connection on  $T^2 \rightarrow S^1$ ,  $(\varphi, \psi) \mapsto \psi$ , whose horizontal space at any point is generated by  $\frac{\partial}{\partial \varphi}$ , by the map which is the restriction of the cotangent bundle projection  $T^*T^2 \rightarrow T^2$  to  $J^{-1}(\mu)$ . Relative to this connection and identifying  $T^*T^2$  with  $TT^2$  using the kinetic energy metric  $ds^2 = d\varphi^2 + d\psi^2$ ,  $\mu$  is the generator of the vertical part of this curve; note  $\left\| \frac{\partial}{\partial \varphi} \right\|^2 = \left\| \frac{\partial}{\partial \psi} \right\|^2 = 1$ . Thus the differential equation in **Step 2** is on the group  $S^1$  and has right hand side given by the generator of the vertical part of the horizontally lifted curve in **Step 1**. Roughly, this describes the method of *reconstruction of dynamics*. We shall explain this in §2 and address the specific case of Lagrangian systems in §3, circumventing the use of the connection in **Step 1**.

### §1D Coupled bodies, linkages and optimal control

The above example can be generalized to the case of coupled rigid bodies. Already the case of a single rigid body in space is an interesting example that will be discussed in §1G below. For several coupled rigid bodies, the dynamics is quite complex. For instance for bodies in the plane, the dynamics is known to be chaotic, despite the presence of stable relative equilibria. See Oh, Sreenath, Krishnaprasad, and Marsden [1989]. Berry phase phenomena for this type of example are quite interesting and are related to some of the work of Wilczek and Shapere on locomotion in micro-organisms. (See, for example, Shapere and Wilczek [1987]). In this problem, control of the system's *internal variables* can lead to phase changes in the *external variables*. These choices of variables are related to the variables in the reduced and the unreduced phase spaces, as we shall see. In this setting one can formulate interesting questions of optimal control such as "when a cat falls and turns itself over in mid flight (all the time with zero angular momentum!) does it do so with optimal efficiency in terms of say energy expended?"



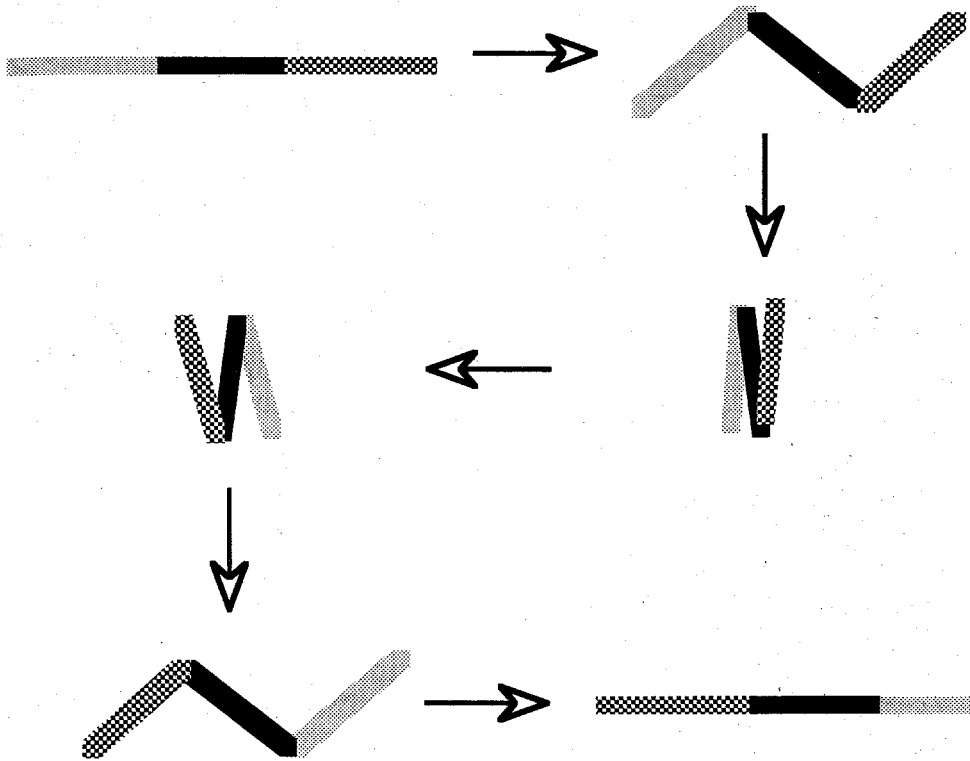


Figure 1D-1

There are interesting answers to these questions that are related to the dynamics of Yang-Mills particles moving in the associated gauge field of the problem. See Montgomery [1989] and references therein. This bundle approach to mechanics will be a theme developed in this work as well.

We give two simple examples of how this works. Additional details will be given for this type of example in §5. First, consider three coupled bars (or coupled planar rigid bodies) linked together with pivot (or pin) joints, so the bars are free to rotate relative to each other. Assume the bars are moving freely on the plane with no external forces and that the angular momentum is zero. However, assume that the joint angles can be controlled with, say, motors in the joints. Figure 1D-1 shows how the joints can be manipulated, each one going through an angle of  $2\pi$  and yet the overall assemblage rotates through an angle  $\pi$ . A formula for the reconstruction phase applicable to examples of this type is given in Krishnaprasad [1989].

A second example is the dynamics of linkages; see §5E for more details. This type of example is considered in Krishnaprasad [1990], Yang and Krishnaprasad [1989], and Krishnaprasad and Yang [1990], including comments on the relation with the three manifold

theory of Thurston. Here one considers a linkage of rods, say four rods linked by pivot joints as in Figure 1D-2. The system is free to rotate without external forces or torques, but there are assumed to be torques at the joints. When one turns the small "crank" the whole assemblage turns even though the angular momentum, as in the previous example, stays zero.

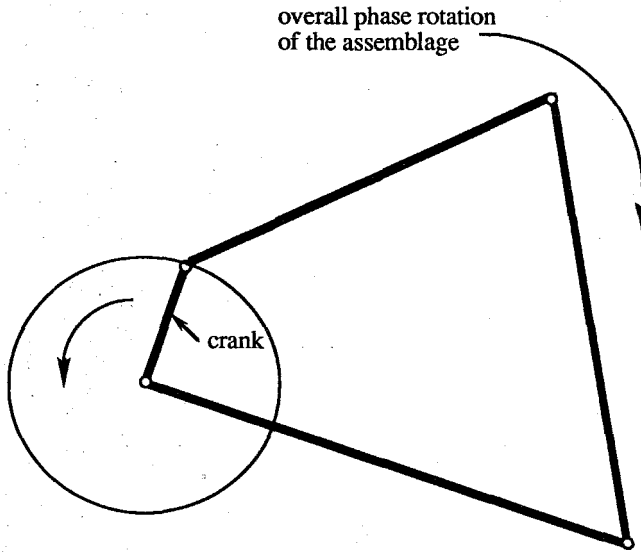


Figure 1D-2

### §1E *Quantum mechanics*

The original motivation for geometric phases came from quantum mechanics. Here the important contributions historically were by Kato in 1950 (for the quantum adiabatic theorem), Longuet-Higgins in 1958 for anomalous spectra in rotating molecules, Berry [1984] who first saw the geometry of the phenomena for a variety of systems, and Simon [1983] who explicitly realized the phases as the holonomy of the Chern-Bott connection. For more information on quantum mechanical phases, and for the references quoted, see the collection of papers in Shapere and Wilczek [1988].

For the purposes of the present work, the paper of Aharonov and Anandan [1987] plays an important role. They got rid of the adiabaticity and showed that the phase for a closed loop in projectivized complex Hilbert space is the exponential of the symplectic area of a two-dimensional manifold whose boundary is the given loop. The symplectic form in question is naturally induced on the projective space from the canonical symplectic form of complex Hilbert space (minus the

imaginary part of the inner product) via reduction, as in Abraham and Marsden [1978]. We shall show in §4 that this formula is the holonomy of the closed loop relative to a principal  $S^1$ -connection on complex Hilbert space and is a particular case of the holonomy formula in principal bundles with abelian structure group.

Littlejohn [1988] has shown that the Bohr-Sommerfeld and Maslov phases of semi-classical mechanics can be viewed as incarnations of Berry's phase. To do this he notes that Gaussian wave-packets define an embedding of classical phase space into Hilbert space, then uses the Aharonov-Anandan point of view on phases, together with the variational formulation of quantum mechanics. The quantum-classical relation between the phases is also considered in Hannay [1985], Anandan [1988], and Weinstein [1989a,b].

### §1F Integrable systems

Consider an integrable system with action-angle variables  $(I_1, I_2, \dots, I_n, \theta_1, \theta_2, \dots, \theta_n)$  and with a Hamiltonian  $H(I_1, I_2, \dots, I_n, \theta_1, \theta_2, \dots, \theta_n; m)$  that depends on a parameter  $m \in M$ . Let  $c$  be a loop based at a point  $m_0$  in  $M$ . We want to compare the angular variables in the torus over  $m_0$ , once the system is slowly changed as the parameters undergo the circuit  $c$ . Since the dynamics in the fiber varies as we move along  $c$ , even if the actions vary by a negligible amount, there will be a shift in the angle variables due to the frequencies  $\omega^i = \partial H / \partial I^i$  of the integrable system; correspondingly, one defines

$$\text{dynamic phase} = \int_0^1 \omega^i(I, c(t)) dt .$$

Here we assume that the loop is contained in a neighborhood whose standard action coordinates are defined. In completing the circuit  $c$ , we return to the same torus, so a comparison between the angles makes sense. The actual shift in the angular variables during the circuit is the dynamic phase plus a correction term called the *geometric phase*. One of the key results is that this geometric phase is the holonomy of an appropriately constructed connection called the *Hannay-Berry connection* on the torus bundle over  $M$  which is constructed from the action-angle variables. The corresponding angular shift, computed by Hannay [1985], is called *Hannay's angles*, so the actual phase shift is given by

$\Delta\theta = \text{dynamic phases} + \text{Hannay's angles} .$
---

The geometric construction of the Hannay-Berry connection for classical systems is given in terms of momentum maps and averaging in Golin, Knauf, and Marmi [1989] and Montgomery [1988].

In this paper we will put this geometry into a more general context and will synthesise it with our work on connections associated with moving systems.

## §1G The Rigid Body

The motion of a rigid body is a geodesic with respect to a left-invariant Riemannian metric (the inertia tensor) on  $\mathbf{SO}(3)$ . The corresponding phase space is  $P = T^*\mathbf{SO}(3)$  and the momentum map  $J : P \rightarrow \mathbb{R}^3$  for the *left*  $\mathbf{SO}(3)$  action is *right* translation to the identity. We identify  $\mathfrak{so}(3)^*$  with  $\mathfrak{so}(3)$  via the Killing form and identify  $\mathbb{R}^3$  with  $\mathfrak{so}(3)$  via the map  $v \mapsto \hat{v}$  where  $\hat{v}(w) = v \times w$ ,  $\times$  being the standard cross product. Points in  $\mathfrak{so}(3)^*$  are regarded as the left reduction of  $T^*\mathbf{SO}(3)$  by  $\mathbf{SO}(3)$  and are the angular momenta as seen from a *body-fixed* frame. The reduced spaces  $J^{-1}(\mu)/G_\mu$  are identified with spheres in  $\mathbb{R}^3$  of Euclidean radius  $\|\mu\|$ , with their symplectic form  $\omega_\mu = -dS/\|\mu\|$  where  $dS$  is the standard area form on a sphere of radius  $\|\mu\|$  and where  $G_\mu$  consists of rotations about the  $\mu$ -axis. The trajectories of the reduced dynamics are obtained by intersecting a family of homothetic ellipsoids (the energy ellipsoids) with the angular momentum spheres. In particular, all but at most four of the reduced trajectories are periodic. These four exceptional trajectories are the well known homoclinic trajectories.

Suppose a reduced trajectory  $\Pi(t)$  is given on  $P_\mu$ , with period  $T$ . After time  $T$ , by how much has the rigid body rotated in space? The spatial angular momentum is  $\pi = \mu = g\Pi$ , which is the conserved value of  $J$ . Here  $g \in \mathbf{SO}(3)$  is the attitude of the rigid body and  $\Pi$  is the body angular momentum. If  $\Pi(0) = \Pi(T)$  then  $\mu = g(0)\Pi(0) = g(T)\Pi(T)$  and so  $g(T)^{-1}\mu = g(0)^{-1}\mu$  i.e.,  $g(T)g(0)^{-1}$  is a rotation about the axis  $\mu$ . We want to compute the angle of this rotation.

To answer this question, let  $c(t)$  be the corresponding trajectory in  $J^{-1}(\mu) \subset P$ . Identify  $T^*\mathbf{SO}(3)$  with  $\mathbf{SO}(3) \times \mathbb{R}^3$  by left trivialization, so  $c(t)$  gets identified with  $(g(t), \Pi(t))$ . Since the reduced trajectory  $\Pi(t)$  closes after time  $T$ , we recover the fact that  $c(T) = gc(0)$  for some  $g \in G_\mu$ . Here,  $g = g(T)g(0)^{-1}$  in the preceding notation. Thus, we can write

$$g = \exp[(\Delta\theta)\zeta] \quad (1)$$

where  $\zeta = \mu/\|\mu\|$  identifies  $\mathfrak{g}_\mu$  with  $\mathbb{R}$  by  $a\zeta \mapsto a$ , for  $a \in \mathbb{R}$ . Let  $D$  be one of the two spherical caps on  $S^2$  enclosed by the reduced trajectory,  $\Lambda$  be the corresponding oriented solid angle, i.e.,  $|\Lambda| = (\text{area } D)/\|\mu\|^2$ , and let  $H_\mu$  be the energy of the reduced trajectory. All norms are taken relative to the Euclidean metric of  $\mathbb{R}^3$ . We shall prove below that modulo  $2\pi$ , we have

$$\Delta\theta = \frac{1}{\|\mu\|} \left\{ \int_D \omega_\mu + 2H_\mu T \right\} = -\Lambda + \frac{2H_\mu T}{\|\mu\|}. \quad (2)$$

The diagram shows a 3D representation of a trajectory in a phase space. The top part is a cylinder-like shape representing the "true (reconstructed) trajectory". A vertical arrow labeled  $\xi(t)$  indicates the "dynamic phase". The horizontal distance between the start and end of the trajectory is labeled "holonomy (geometric phase)". A vertical arrow labeled  $\pi_\mu$  points down to a shaded sphere representing the "reduced trajectory". The sphere is labeled  $P_\mu$  and has a point  $D$  on its surface. A curved arrow on the sphere indicates the "reduced trajectory". A label  $\ln J^{-1}(\mu)$  with two arrows points towards the sphere, indicating its relationship to the holonomy.

To prove (2), we choose the connection one-form on  $J^{-1}(\mu)$  to be (see Proposition 2.2)

where  $\theta_\mu$  is the pull back to  $J^{-1}(\mu)$  of the canonical one-form  $\theta$  on  $T^*\mathbf{SO}(3)$ . The curvature of  $A$  as a two-form on the base  $P_\mu$ , the sphere of radius  $\|\mu\|$  in  $\mathbb{R}^3$ , is given by

$$-\frac{1}{\|\mu\|} \omega_\mu = \frac{1}{\|\mu\|^2} dS. \quad (4)$$

The first terms in (2) represent the *geometric phase*, i.e., the holonomy of the reduced trajectory with respect to this connection. By Corollary 4.2, the logarithm of the holonomy (modulo  $2\pi$ ) is given as minus the integral over  $D$  of the curvature, i.e., it equals

$$\frac{1}{\|\mu\|} \int_D \omega_\mu = -\frac{1}{\|\mu\|^2} (\text{area } D) = -\Lambda \pmod{2\pi} \quad (5)$$

The second terms in (2) represent the dynamic phase. By the algorithm of Proposition 2.1 it is calculated in the following way. First one horizontally lifts the reduced closed trajectory  $\Pi(t)$  to  $J^{-1}(\mu)$  relative to the connection (3). This horizontal lift is easily seen to be (identity,  $\Pi(t)$ ) in the left trivialization of  $T^*\mathbf{SO}(3)$  as  $\mathbf{SO}(3) \times \mathbb{R}^3$ . Second, one computes

$$\xi(t) = (A \cdot X_H)(\Pi(t)). \quad (6)$$

Since in coordinates

$$\theta_\mu = \sum_i p_i dq^i \text{ and } X_H = \sum_i p^i \frac{\partial}{\partial q^i} + \frac{\partial}{\partial p} \text{ terms}$$

for  $p^i = \sum_j g^{ij} p_j$ ,  $g^{ij}$  being the inverse of the Riemannian metric  $g_{ij}$  on  $\mathbf{SO}(3)$ , we get

$$(\theta_\mu \cdot X_H)(\Pi(t)) = \sum_i p_i p^i = 2H(\text{identity}, \Pi(t)) = 2H_\mu. \quad (7)$$

where  $H_\mu$  is the value of the energy on  $S^2$  along the integral curve  $\Pi(t)$ . Consequently,

$$\xi(t) = \frac{2H_\mu}{\|\mu\|} \zeta \quad (8)$$

Third, since  $\xi(t)$  is independent of  $t$ , the solution of the equation

$$\dot{g} = g\xi = \frac{2H_\mu}{\|\mu\|} g\zeta \quad \text{is} \quad g(t) = \exp\left(\frac{2H_\mu t}{\|\mu\|} \zeta\right)$$

so that the dynamic phase equals

$$\Delta\theta_d = \frac{2H_\mu}{\|\mu\|} T \pmod{2\pi} \quad (9)$$

Formulas (5) and (9) prove (2). Note that (2) is independent of which spherical cap one chooses amongst the two bounded by  $\Pi(t)$ . Indeed, the solid angles on the unit sphere defined by the two caps add to  $4\pi$ , which does not change formula (2).

## §2 Reconstruction of Dynamics for Hamiltonian Systems

This section presents a reconstruction method for the dynamics of a given Hamiltonian system from that of the reduced system. The method and formulas found here are applied in the next section to Lagrangian systems.

### §2A General Considerations

We begin with the abstract reconstruction method. Let  $P$  be a Poisson manifold on which a Lie group acts in a Hamiltonian manner and has a momentum map  $J : P \rightarrow \mathfrak{g}^*$ ; here  $\mathfrak{g}$  is the Lie algebra of  $G$  and  $\mathfrak{g}^*$  is its dual. For a weakly regular value  $\mu \in \mathfrak{g}^*$  of  $J$ , assuming that the *reduced space*  $P_\mu := J^{-1}(\mu)/G_\mu$  is a smooth manifold with the canonical projection a surjective submersion,  $P_\mu$  a Poisson manifold (see Marsden and Ratiu [1986] for the general theory of Poisson reduction). Given  $f, h : P_\mu \rightarrow \mathbb{R}$ , lift them to  $J^{-1}(\mu)$  by  $\pi_\mu$ , then extend them to  $G$ -invariant functions on  $J^{-1}(O_\mu)$ , where  $O_\mu$  is the coadjoint orbit of  $\mu$  in  $\mathfrak{g}^*$ , and then extend these functions arbitrarily to  $\bar{f}, \bar{h} : P \rightarrow \mathbb{R}$ . The Poisson bracket of  $f$  and  $h$  in the Poisson structure of  $P_\mu$  is defined by  $\{f, h\} \circ \pi_\mu = \{\bar{f}|_{O_\mu}, \bar{h}|_{O_\mu}\}$ . If  $P$  is symplectic, then  $P_\mu$  is also symplectic (see Marsden and Weinstein [1974] and Abraham and Marsden [1978], Chapter 4). If  $H : P \rightarrow \mathbb{R}$  is a  $G$ -invariant Hamiltonian it induces a Hamiltonian  $H_\mu : P_\mu \rightarrow \mathbb{R}$  and the flow of the Hamiltonian vector field  $X_{H_\mu}$  on  $P_\mu$  is the  $G_\mu$ -quotient of the flow of  $X_H$  on  $J^{-1}(\mu)$ .

Assume that an integral curve  $c_\mu(t)$  of  $X_{H_\mu}$  on  $P_\mu$  is known. For  $p_0 \in J^{-1}(\mu)$ , we search for the corresponding integral curve  $c(t) = F_t(p_0)$  of  $X_H$  such that  $\pi_\mu(c(t)) = c_\mu(t)$ , where  $\pi_\mu : J^{-1}(\mu) \rightarrow P_\mu$  is the projection. Pick a smooth curve  $d(t)$  in  $J^{-1}(\mu)$  such that  $d(0) = p_0$  and  $\pi_\mu(d(t)) = c_\mu(t)$ . Write  $c(t) = \Phi_{g(t)}(d(t))$  for some curve  $g(t)$  in  $G_\mu$  to be determined. We have

$$X_H(c(t)) = c'(t) = T_{d(t)}\Phi_{g(t)}(d'(t)) + T_{d(t)}\Phi_{g(t)} \cdot \left( T_{g(t)}L_{g(t)^{-1}}(g'(t)) \right)_P(d(t)). \quad (1)$$

Since  $\Phi_g^* X_H = X_{\Phi_g^* H} = X_{H_\mu}$ , (1) gives

$$d'(t) + \left( T_{g(t)}L_{g(t)^{-1}}(g'(t)) \right)_P(d(t)) = T\Phi_{g(t)^{-1}}X_H(\Phi_{g(t)}(d(t))) = (\Phi_{g(t)}^* X_H)(d(t)) = X_{H_\mu}(d(t)). \quad (2)$$

This is an equation for  $g(t)$  written in terms of  $d(t)$  only. We solve it in two steps:

**Step 1** Find  $\xi(t) \in \mathfrak{g}_\mu$  such that

$$\xi(t)_P(d(t)) = X_H(d(t)) - d'(t); \quad (3)$$

**Step 2** With  $\xi(t)$  determined, solve the non-autonomous ordinary differential equation on  $G_\mu$ :

$$g'(t) = T_e L_{g(t)}(\xi(t)), \quad \text{with } g(0) = e. \quad (4)$$

Step 1 is typically of an algebraic nature; in coordinates, for matrix Lie groups, (3) is just a matrix equation. We show later how  $\xi(t)$  can be explicitly computed if a connection is given on  $J^{-1}(\mu) \rightarrow P_\mu$ . Step 2 gives an answer "in quadratures" if  $G$  is abelian as we shall see in formula (5) below. In general (4) cannot be solved explicitly and represents the main technical difficulty in the reconstruction method. With  $g(t)$  determined, the desired integral curve  $c(t)$  is given by  $c(t) = \Phi_{g(t)}(d(t))$ . The same construction works on  $P/G$ , even if the  $G$ -action does not admit a momentum map.

Step 2 can be carried out *explicitly* when  $G$  is abelian. Here the connected component of the identity of  $G$  is a cylinder  $\mathbb{R}^p \times \mathbb{T}^{k-p}$  and the exponential map  $\exp(\xi_1, \dots, \xi_p, \xi_{p+1}(\bmod 2\pi), \dots, \xi_k(\bmod 2\pi))$  is onto, so we can write  $g(t) = \exp \eta(t)$ ,  $\eta(0) = 0$ . Therefore  $\xi(t) = T_{g(t)} L_{g(t)^{-1}}(g'(t)) = \eta'(t)$  since  $\eta'$  and  $\eta$  commute, i.e.,  $\eta(t) = \int_0^t \xi(s) ds$ . Thus the solution of (4) in Step 2 when  $G$  is abelian is

$$g(t) = \exp \left( \int_0^t \xi(s) ds \right) \quad (5)$$

This reconstruction method depends on the choice of  $d(t)$ . With additional structure,  $d(t)$  can be chosen in a *natural geometric* way. What is needed is a way of lifting curves on the base of a principal bundle to curves in the total space. We do this using connections. One can object at this point that at the moment, reconstruction involves integrating one ordinary differential equation, whereas introducing a connection will involve integration of *two* ordinary differential equations, one for the horizontal lift and one for constructing the solution of (4) from it. However, for the determination of phases, there are some situations in which the phase can be computed without actually solving either equation, so one actually solves *no* differential equations; a specific case is the rigid body, discussed in §1G. In other circumstances, one can compute the horizontal lift explicitly (see Marsden, Ratiu, and Raugel [1990]). However, in general, without such added information, it is true that the number of equations in principle is two rather than one.

Suppose that  $\pi_\mu : J^{-1}(\mu) \rightarrow P_\mu$  is a principal  $G_\mu$ -bundle with a *connection*  $A$ . This means that  $A$  is a  $\mathfrak{g}_\mu$ -valued one-form on  $J^{-1}(\mu) \subset P$  satisfying



- i  $A_p \cdot \xi_p(p) = \xi$  for  $\xi \in \mathfrak{g}_\mu$
- ii  $L_g^* A = \text{Ad}_g \circ A$  for  $g \in G_\mu$ .

Let  $d(t)$  be the *horizontal lift* of  $c_\mu$  through  $p_0$ ; i.e.,  $A \cdot d'(t) = 0$ ,  $\pi_\mu \circ d = c_\mu$ , and  $d(0) = p_0$ .

**2.1 Theorem** Suppose  $\pi_\mu : J^{-1}(\mu) \rightarrow P_\mu$  is a principal  $G_\mu$ -bundle with connection  $A$ . Let  $c_\mu$  be an integral curve of the reduced dynamical system on  $P_\mu$ . Then the corresponding curve  $c$  (through  $p_0 \in \pi_\mu^{-1}(c_\mu(0))$ ) of the system on  $P$  is determined as follows:

- i Horizontally lift  $c_\mu$  to form the curve  $d$  in  $J^{-1}(\mu)$  through  $p_0$ .
- ii Set  $\xi(t) = A \cdot X_H(d(t))$ , so that  $\xi(t)$  is a curve in  $\mathfrak{g}_\mu$ .
- iii Solve the equation  $\dot{g}(t) = g(t) \cdot \xi(t)$ .

Then  $c(t) = g(t) \cdot d(t)$  is the integral curve of the system on  $P$  with initial condition  $p_0$ .

Suppose  $c_\mu$  is a closed curve; thus, both  $c$  and  $d$  reintersect the same fiber. Write

$$d(1) = g \cdot d(0) \quad \text{and} \quad c(1) = h \cdot c(0)$$

for  $g, h \in G_\mu$ . Note that

$$h = g(1) \cdot g. \quad (6)$$

The Lie group element  $g$  (or the Lie algebra element  $\log g$ ) is called the *geometric phase*. It is the holonomy of the path  $c_\mu$  with respect to the connection  $A$  and has the important property of being parametrization independent. The Lie group element  $g(1)$  (or  $\log g(1)$ ) is called the *dynamic phase*.

For compact or semi-simple  $G$ ,  $G_\mu$  is generically abelian. The computation of  $g(1)$  and  $g$  are then relatively easy, as was indicated above.

## §2B Cotangent Bundle with $G_\mu$ one Dimensional

We now discuss the case in which  $P = T^*Q$ , and  $G$  acts on  $Q$  and therefore on  $P$  by cotangent lift. In this case the momentum map is given by the formula

$$J(\alpha_q) \cdot \xi = \alpha_q \cdot \xi_Q(q) = \xi_{T^*Q}(\alpha_q) \lrcorner \theta(\alpha_q),$$

where  $\xi \in \mathfrak{g}_\mu$ ,  $\alpha_q \in T_q^*Q$ ,  $\theta = \sum_i p_i dq^i$  is the canonical one-form and  $\lrcorner$  is the interior product.

Assume  $G_\mu$  is the circle group or the line. Pick a generator  $\zeta \in \mathfrak{g}_\mu$ ,  $\zeta \neq 0$ . For instance, one can choose the shortest  $\zeta$  such that  $\exp(2\pi\zeta) = 1$ . Identify  $\mathfrak{g}_\mu$  with the real line via  $\omega \mapsto \omega\zeta$ . Then a connection one-form is a standard one-form on  $J^{-1}(\mu)$ .

**2.2 Proposition** Suppose  $G_\mu \cong S^1$  or  $\mathbb{R}$ . Identify  $\mathfrak{g}_\mu$  with  $\mathbb{R}$  via a choice of generator  $\zeta$ . Let  $\theta_\mu$  denote the pull-back of the canonical one-form to  $J^{-1}(\mu)$ . Then

$$A = \frac{1}{\langle \mu, \zeta \rangle} \theta_\mu \otimes \zeta$$

is a connection one-form on  $J^{-1}(\mu) \rightarrow P_\mu$ . Its curvature as a two-form on the base  $P_\mu$  is

$$\Omega = -\frac{1}{\langle \mu, \zeta \rangle} \omega_\mu,$$

where  $\omega_\mu$  is the reduced symplectic form on  $P_\mu$ .

**Proof** Since  $G$  acts by cotangent lift, it preserves  $\theta$ , and so  $\theta_\mu$  is preserved by  $G_\mu$  and therefore  $A$  is  $G_\mu$ -invariant. Also,  $A \cdot \zeta_p = [\zeta_p \lrcorner \theta / \langle \mu, \zeta \rangle] \zeta = [J^\zeta / \langle \mu, \zeta \rangle] \zeta = \zeta$ . This verifies that  $A$  is a connection. The calculation of its curvature is straightforward. (See §4 and note that  $\omega = -d\theta$  in our conventions.) ■

**Remarks 1** The result of Proposition 2.2 holds for any *exact* symplectic manifold. We shall use this in §5A.

**2** In the next section we shall show how to construct a connection on  $J^{-1}(\mu) \rightarrow P_\mu$  in general. For the case  $Q = G$  it includes the connection in 2.2. For  $Q = \mathfrak{SO}(3)$  this recovers the connection for the rigid body. We will return to this point shortly.

## §2C Cotangent Bundles - General Case

If  $G_\mu$  is not abelian, the formula for  $A$  given above does not satisfy the second axiom of a connection. However, if the bundle  $Q \rightarrow Q/G_\mu$  has a connection, we will show below how this induces a connection on  $J^{-1}(\mu) \rightarrow (T^*Q)_\mu$ . To do this, we recall the cotangent bundle reduction theorem of Satzer, Marsden and Kummer (see Abraham and Marsden [1978], §4.3 and Kummer [1981]).

Assume the Lie group  $G$  acts freely on the left on  $Q$ ; lift this to a symplectic action on  $T^*Q$ . The momentum map of this lift is  $J(\alpha_q) \cdot \xi = \alpha_q \cdot \xi_Q(q)$ , where  $\alpha_q \in T_q^*Q$ ,  $\xi \in \mathfrak{g}$ , and

$\xi_Q(q) = \left. \frac{d}{dt} \right|_{t=0} (\exp t\xi \cdot q)$  is the infinitesimal generator of the  $G$ -action on  $Q$  defined by  $\xi$ .

Assume that  $\mu$  is a weakly regular value and that  $J^{-1}(\mu)/G_\mu = (T^*Q)_\mu$  is a smooth manifold and that the canonical projection  $\pi_\mu : J^{-1}(\mu) \rightarrow (T^*Q)_\mu$  is a surjective submersion. Let  $\mathfrak{g}_\mu = \{\eta \in \mathfrak{g} \mid (\text{ad } \eta)^* \mu = 0\}$  be the isotropy Lie algebra at  $\mu$ ; i.e., the Lie algebra of  $G_\mu$ , and denote by  $\mu' = (\mu \mid \mathfrak{g}_\mu) \in \mathfrak{g}_\mu^*$ , the restriction of  $\mu$  to  $\mathfrak{g}_\mu$ . Assume that  $\rho_\mu : Q \rightarrow Q/G_\mu$  is a principal bundle and  $\gamma \in \Omega^1(Q; \mathfrak{g}_\mu)$  is a left connection one-form on  $Q$ , i.e.,  $\gamma(\eta_Q) = \eta$  for all  $\eta \in \mathfrak{g}_\mu$  and  $\gamma$  is  $G_\mu$ -equivariant:  $\gamma_{g \cdot q}(g \cdot v) = \text{Ad}_g(\gamma(v))$  for all  $g \in G_\mu$ ,  $q \in Q$ ,  $v \in T_q Q$ .

**2.3 Theorem** *Let  $\text{curv}(\gamma)$  be the curvature of  $\gamma$  and let  $B$  be the pull-back by the cotangent bundle projection  $T^*(Q/G_\mu) \rightarrow Q/G_\mu$  of the two form on  $Q/G_\mu$  induced by the  $\mu'$ -component  $\mu' \cdot \text{curv}(\gamma) \in \Omega^2(Q)$  of  $\text{curv}(\gamma)$ ; thus  $\mu' \cdot \text{curv}(\gamma)$  is a closed real-valued two-form on  $Q$  and  $B$  is a closed two-form on  $T^*(Q/G_\mu)$ . Endow  $T^*(Q/G_\mu)$  with the symplectic form  $\omega - B$ , where  $\omega$  is the canonical two-form of the cotangent bundle. Then  $(T^*Q)_\mu$  is symplectically embedded in  $(T^*(Q/G_\mu), \omega - B)$  and its image is a vector subbundle with base  $Q/G_\mu$ . This embedding is onto if and only if  $\mathfrak{g} = \mathfrak{g}_\mu$ .*

Denote by  $J_\mu : T^*Q \rightarrow \mathfrak{g}_\mu^*$  the induced momentum map, i.e.,  $J_\mu(\alpha_q) = J(\alpha_q) \mid \mathfrak{g}_\mu$ . From the proof of the theorem, which is ultimately based on a momentum shift in the fiber, it follows that this diagram commutes:

$$\begin{array}{ccccccc}
 J^{-1}(\mu) & \xrightarrow{\quad} & J_\mu^{-1}(\mu') & \xrightarrow{t_\mu} & J^{-1}(0) & \xrightarrow{\quad} & T^*Q \xrightarrow{\pi} Q \\
 \downarrow \pi_\mu & & & & \downarrow & & \downarrow \rho_\mu \\
 (T^*Q)_\mu & \xrightarrow{\quad} & T^*(Q/G_\mu) & \xrightarrow{\quad} & & \xrightarrow{\quad} & Q/G_\mu \\
 [\alpha_q] & \mapsto & [\alpha_q - \mu' \cdot \gamma_q(\cdot)] & \mapsto & & \mapsto & [q]
 \end{array}$$

where  $t_\mu(\alpha_q) = \alpha_q - \mu' \cdot \gamma_q(\cdot)$  is fiber translation by the  $\mu'$ -component of the connection form and where  $[\alpha_q - \mu' \cdot \gamma_q(\cdot)]$  means the element of  $T^*(Q/G_\mu)$  determined by  $\alpha_q - \mu' \cdot \gamma_q(\cdot)$ . Call the composition of the two maps on the bottom of this diagram  $\sigma : [\alpha_q] \in (T^*Q)_\mu \mapsto [q] \in Q/G_\mu$ . A

natural way of inducing a connection on  $J^{-1}(\mu) \rightarrow (T^*Q)_\mu$  is to be consistent with this theorem, i.e., by pull-back in the diagram above.

**2.4 Corollary** *The connection one-form  $\gamma \in \Omega^1(Q; \mathfrak{g}_\mu)$  induces a connection one-form  $\tilde{\gamma}^\mu \in \Omega^1(J^{-1}(\mu); \mathfrak{g}_\mu)$  by pull-back:  $\tilde{\gamma}^\mu = (\pi \circ t_\mu)^* \gamma$ , i.e.,*

$$\tilde{\gamma}^\mu(\alpha_q) \cdot U_{\alpha_q} = \gamma(q) \cdot T_{\alpha_q} \pi(U_{\alpha_q}), \text{ for } \alpha_q \in T_q^*Q, U_{\alpha_q} \in T_{\alpha_q}(T^*Q).$$

Similarly  $\text{curv}(\tilde{\gamma}^\mu) = (\pi \circ t_\mu)^* \text{curv}(\gamma)$  and in particular the  $\mu'$ -component of the curvature of this connection equals  $B$ , the pull-back of  $\mu'$ -curv( $\gamma$ ).

The proof is a direct verification.

**2.5 Corollary** *Assume that  $\rho_\mu : Q \rightarrow Q/G_\mu$  is a principal  $G_\mu$ -bundle with a connection  $\gamma \in \Omega^1(Q; \mathfrak{g}_\mu)$ . If  $H$  is a  $G$ -invariant Hamiltonian on  $T^*Q$  inducing the Hamiltonian  $H_\mu$  on  $(T^*Q)_\mu$  and  $c_G(t)$  is an integral curve of  $X_{H_\mu}$ , denote by  $d(t)$  a horizontal lift of  $c_G(t)$  in  $J^{-1}(\mu)$  relative to the natural connection of Corollary 2.4 and let  $q(t) = \pi(d(t))$  be the base integral curve of  $c(t)$ . Then  $\xi(t)$  of step ii in Theorem 2.1 is given by*

$$\xi(t) = \gamma(q(t)) \cdot FH(d(t)),$$

where  $FH : T^*Q \rightarrow TQ$  is the fiber derivative of  $H$ , i.e.,  $FH(\alpha_q) \cdot \beta_q = \left. \frac{d}{dt} \right|_{t=0} H(\alpha_q + t\beta_q)$ .

**Proof**  $\tilde{\gamma}^\mu \cdot X_H = \gamma \cdot T\pi \cdot X_H = \gamma \cdot FH$ . ■

If  $Q$  carries a  $G$ -invariant Riemannian metric  $\langle\langle \cdot, \cdot \rangle\rangle$  define an associated connection  $\gamma_{\text{mech}} \in \Omega^1(Q; \mathfrak{g}_\mu)$  by declaring the horizontal space at any  $q \in Q$  to be the orthogonal complement of the vertical space. Assume also that the Hamiltonian  $H$  is of the form kinetic energy with respect to the metric  $\langle\langle \cdot, \cdot \rangle\rangle$  plus a  $G$ -invariant potential energy.

**2.6 Corollary** *Under these hypotheses, step ii in Proposition 2.1 is equivalent to*  
ii'  $\xi(t) \in \mathfrak{g}_\mu$  *is given by*

$$\xi(t) = \gamma_{\text{mech}}(q(t)) \cdot d(t)^\#, \text{ where } d(t) \in T_{q(t)}^*Q.$$

**Proof** Apply Corollary 2.4 and use the fact that  $FH(\alpha_q) = \alpha_q^\#$ . (In coordinates,  $\frac{\partial H}{\partial p_\mu} = g^{\mu\nu} p_\nu$ ).

■

In this case,  $\gamma_{\text{mech}}$  is closely related to the *mechanical connection* that appears in the work of Smale [1970], Guichardet [1984], Wilczek and Shapere [1988], Montgomery [1988,90a,b], Lewis, Marsden, Simo and Posbergh [1989] and Simo, Lewis and Marsden [1989]. It is given explicitly as follows: let  $\mathbb{I}_\mu(q): \mathfrak{g}_\mu \rightarrow \mathfrak{g}_\mu^*$  be defined by  $\mathbb{I}_\mu(\zeta)(\eta) = \langle \zeta_Q(q), \eta_Q(q) \rangle$  be the  $\mu$ -*locked inertia tensor*. This name is used because for coupled rigid (or rigid-flexible) structures,  $\mathbb{I}_\mu(q)$  is the inertia tensor of the system with the joints locked in the configuration  $q$ , thereby forming a rigid body. Then  $\gamma_{\text{mech}}: TQ \rightarrow \mathfrak{g}_\mu$  is

$$\gamma_{\text{mech}}(v_q) = \mathbb{I}_\mu(q)^{-1} J(v_q^b)$$

where  $v_q^b$  is the one-form corresponding to the vector  $v_q$  via the metric. Shapere and Wilczek call this the "master formula".

Besides choosing the connection  $\gamma_{\text{mech}}$  to be defined by the *metric* orthogonal to the  $G_\mu$ -orbits, one can use other complements. Here is one such:

**2.7 Corollary** If  $Q = G$ ,  $\dim G_\mu = 1$ , and  $\zeta$  is a generator of  $\mathfrak{g}_\mu$ , the one-form  $\gamma_R \in \Omega^1(G)$  given by

$$\gamma_R(g) = \frac{1}{\langle \mu, \zeta \rangle} T_g^* R_{g^{-1}}(\mu)$$

induces via the procedure in Corollary 2.4 the connection  $A \in \Omega^1(J_L^{-1}(\mu))$  given in Proposition 2.2, where  $J_L(\alpha_g) = T_e^* R_g(\alpha_g)$ . Here,  $\langle \mu, \zeta \rangle$  is the natural pairing between  $\mu$  and  $\zeta$ .

**Proof** The axioms of a left principal  $G_\mu$ -connection, namely  $\gamma_R(\zeta_G) = \zeta$  and  $\gamma_R(hg)(T_g L_h(v_g)) = \gamma_R(g)(v_g)$  for all  $h \in G_g$  and  $v_g \in T_g G$  are straightforward-verifications; recall that  $\zeta_G(g) = T_e R_g(\zeta)$ . Moreover, for  $\alpha_g \in J_L^{-1}(\mu)$  and  $U_{\alpha_g} \in T_{\alpha_g}(J_L^{-1}(\mu))$ , we have

$$\begin{aligned} \tilde{\gamma}_R^{\mu}(\alpha_g) \cdot U_{\alpha_g} &= \gamma_R(g) \cdot T_{\alpha_g} \pi(U_{\alpha_g}) \\ &= \frac{1}{\langle \mu, \zeta \rangle} T_g^* R_{g^{-1}}(\mu) \cdot T_{\alpha_g} \pi(U_{\alpha_g}) = A(\alpha_g) \cdot U_{\alpha_g} \end{aligned}$$

since  $\alpha_g \in J_L^{-1}(\mu)$  is necessarily equal to  $T_g^* R_{g^{-1}}(\mu)$ . ■

### §3 Reconstruction of Dynamics for Lagrangian Systems

In this section we reconstruct the dynamics of a given Lagrangian system with symmetry from the reduced dynamics. We begin by recalling the basic facts about Lagrangian systems.

#### §3A Lagrangian Systems

If  $Q$  is a manifold,  $L : TQ \rightarrow \mathbb{R}$  is a smooth function, and  $\tau : TQ \rightarrow Q$  the projection, let  $\mathbb{F}L : TQ \rightarrow T^*Q$  be the *fiber derivative* of  $L$  given by

$$\mathbb{F}L(v) \cdot w = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} L(v + \varepsilon w),$$

for  $v, w \in T_q Q$ . If  $\Omega$  denotes the canonical symplectic structure on  $T^*Q$ , let  $\Omega_L = (\mathbb{F}L)^* \Omega$ , a closed two form on  $TQ$ . If  $\Omega_L$  is a (weak) symplectic form, we call  $L$  *regular*. In local charts this is equivalent to the second derivative in the fiber variable to be a (weakly) non-degenerate bilinear form. If  $\mathbb{F}L$  is a diffeomorphism,  $L$  is called *hyperregular* and  $\mathbb{F}L$  the *Legendre transformation*. Returning to a general Lagrangian  $L$ , let  $A(v) = \mathbb{F}L(v) \cdot v$  be the *action* of  $L$  and  $E = A - L$  the *energy* of  $L$ . A vector field  $Z \in \mathfrak{X}(TQ)$  is called a *Lagrangian system*, if  $i_Z \Omega_L = dE$ , where  $i_Z$  denotes the interior product (or contraction) with  $Z$ . If  $L$  is regular, then  $Z$  is a second order equation, i.e.,  $T\tau \circ Z = \text{identity on } TQ$ , where  $\tau : TQ \rightarrow Q$  is the tangent bundle projection. In general, if we assume that  $Z$  is second order, then locally Lagrange's equations hold:

$$\frac{d}{dt}(D_2 L(q(t), \dot{q}(t))) = D_1 L(q(t), \dot{q}(t))$$

where  $\dot{q}(t) = dq(t)/dt$  and  $D_1$  and  $D_2$  denote the partial Fréchet derivatives. Moreover,  $E$  is conserved by the flow of  $Z$ . In this section we will assume that  $Z$  is a second order equation.

If  $L$  is hyperregular, then  $Z = X_E$  is the Hamiltonian vector field relative to  $\Omega_L$  defined by  $E$ . Then  $H = E \circ (\mathbb{F}L)^{-1} : T^*Q \rightarrow \mathbb{R}$  defines a Hamiltonian system  $X_H$  on  $T^*Q$  whose flow is conjugate by  $\mathbb{F}L$  to that of  $X_E$  on  $TQ$ . The fiber derivative  $\mathbb{F}H : T^*Q \rightarrow TQ$  of  $H$  (defined in Corollary 2.5) is the inverse of  $\mathbb{F}L$ . Conversely, if  $H : T^*Q \rightarrow \mathbb{R}$  is hyperregular, i.e.,  $\mathbb{F}H : T^*Q \rightarrow TQ$  is diffeomorphism, let  $\Theta(X_H)$  be the *action* of  $H$ , where  $\Theta$  is the canonical one-form on  $T^*Q$ . Our convention is  $\Omega = -d\Theta$ . Let  $A = \Theta(X_H) \circ (\mathbb{F}H)^{-1}$ ,  $E = H \circ (\mathbb{F}H)^{-1}$ , and  $L =$

A – E. Then  $L$  is the Lagrangian system inducing  $H$  and the above prescription defines a bijective correspondence between hyperregular Lagrangians on  $TQ$  and hyperregular Hamiltonians on  $T^*Q$ . Moreover, the base integral curves (*i.e.*, the projections of the integrals curves on  $TQ$  and  $T^*Q$  onto  $Q$ ) of  $X_E$  and  $X_H$  coincide. See, *e.g.*, Abraham and Marsden [1978] for the proofs.

Let a Lie group  $G$  act on  $Q$  and that  $L$  is invariant under the lifted  $G$ -action to  $TQ$ . The Legendre transformation  $FL$  is equivariant relative to this action and the cotangent lifted action on  $T^*Q$ , and so  $A, E, \Omega_L$ , and  $Z$  are  $G$ -invariant. Assuming that  $(TQ)/G$  is a smooth manifold with the projection  $\rho : TQ \rightarrow (TQ)/G$  a surjective submersion,  $E$  induces a smooth function  $E_G$  and  $Z$  a smooth vector field  $Z_G$  on  $(TQ)/G$ ; the flow of  $Z_G$  conserves  $E_G$ . The question we shall address in this section is the following: given an integral curve  $c_G(t)$  of  $Z_G$ ,  $c_G(0) = [v_q]$  and  $v_q \in T_q Q$ , construct the integral curve  $c(t)$  of  $Z$  satisfying  $c(0) = v_q$ .

### §3B Reconstruction for $Q = G$

We begin with the simplest case:  $Q = G$  with a left-invariant Lagrangian  $L : TG \rightarrow \mathbb{R}$ ,  $L(T_g L_h(v_g)) = L(v_g)$ , where  $h, g \in G$ ,  $v_g \in T_g G$ ,  $L_h : G \rightarrow G$  denotes left-translation by  $h$ ,  $L_h(k) = hk$ , and  $T_g L_h : T_g G \rightarrow T_{hg} G$  is its tangent map.

**3.1 Proposition** *Let  $L : TG \rightarrow \mathbb{R}$  be a left-invariant Lagrangian such that its Lagrangian vector field  $Z \in \mathfrak{X}(TG)$  is a second order equation. Let  $Z_G \in \mathfrak{X}(\mathfrak{g})$  be the induced vector field on  $(TG)/G \approx \mathfrak{g}$  and let  $\xi(t)$  be an integral curve of  $Z_G$ . If  $g(t) \in G$  is the solution of the non-autonomous ordinary differential equation  $\dot{g}(t) = T_e L_{g(t)} \xi(t)$ ,  $g(0) = e$ , and  $g \in G$  then  $v(t) = T_e L_{gg(t)} \xi(t)$  is the integral curve of  $Z$  satisfying  $v(0) = T_e L_g \xi(0)$  and  $v(t)$  projects to  $\xi(t)$ , *i.e.*,  $TL_{\tau(v(t))^{-1}} v(t) = \xi(t)$ .*

**Proof** Let  $v(t)$  be the integral curve of  $Z$  satisfying  $v(0) = T_e L_g \xi(0)$  for a given element  $\xi(0) \in \mathfrak{g}$ . Since  $\xi(t)$  is the integral curve of  $Z_G$  whose flow is conjugated to the flow of  $Z$  by left-translation, we have  $TL_{\tau(v(t))^{-1}} v(t) = \xi(t)$ . If  $h(t) = \tau(v(t))$ , since  $Z$  is a second order equation, we have

$$v(t) = \dot{h}(t) = T_e L_{h(t)} \xi(t) \quad \text{and} \quad h(0) = \tau(v(0)) = g$$

so that letting  $g(t) = g^{-1}h(t)$  we get  $g(0) = e$  and

$$\dot{g}(t) = TL_{g^{-1}} \dot{h}(t) = TL_{g^{-1}} TL_{h(t)} \xi(t) = TL_{g(t)} \xi(t).$$

This determines  $g(t)$  uniquely from  $\xi(t)$  and so

$$v(t) = T_e L_{h(t)} \xi(t) = T_e L_{gg(t)} \xi(t). \blacksquare$$

In general,  $Z_G$  is not Hamiltonian. However, if  $L$  is hyperregular, then the Legendre transformation  $\mathbb{F}L$  will induce a Poisson structure on  $\mathfrak{g}$ . In fact, in this case, we can reconstruct the dynamics of the induced Hamiltonian system.

**3.2 Corollary** *Let  $L : TG \rightarrow \mathbb{R}$  be a left-invariant hyperregular Lagrangian,  $H : T^*G \rightarrow \mathbb{R}$  the induced Hamiltonian, and  $h : \mathfrak{g}^* \rightarrow \mathbb{R}$  the induced Hamiltonian on the dual  $\mathfrak{g}^*$  of  $\mathfrak{g}$  endowed with the Lie-Poisson structure;  $h = H|_{\mathfrak{g}^*}$ . Let  $\mu(t)$  be an integral curve of  $X_h(\mu) = \text{ad} \left( \frac{\delta h}{\delta \mu} \right)^* \mu$ ,  $\mu(0) = T_e^* L_g(\alpha_g)$ ,  $\alpha_g \in G$  fixed. If  $\xi(t) = (\mathbb{F}L)^{-1} \mu(t)$  and  $g(t)$  is the solution of  $\dot{g}(t) = T_e L_{g(t)} \xi(t)$ ,  $g(0) = e$ , then  $\alpha(t) = T_e^* L_{(gg(t))^{-1}} \mu(t)$  is the integral curve of  $X_H$  on  $T^*G$ ,  $\alpha(0) = \alpha_g$  and  $\alpha(t)$  projects to  $\mu(t)$ , i.e.,  $T_e^* L_{\pi(\alpha(t))} = \mu(t)$ ;  $\pi : T^*G \rightarrow G$  denotes the cotangent bundle projection.*

The proof is a consequence of the previous result and the fact that  $\mathbb{F}L$  commutes with left translation:  $\mathbb{F}L \circ TL_g = T_e^* L_{g^{-1}} \circ \mathbb{F}L$ .

### §3C General Q

In §3B we studied reconstruction for the case of Lie-Poisson reduction for a given invariant Lagrangian. In the general case, instead we take a symplectic reduction viewpoint. Given is  $L : TQ \rightarrow \mathbb{R}$ ,  $G$ -invariant,  $Z \in \mathfrak{X}((TQ))$  its Lagrangian vector field which we assume to be a second order equation. The lift of the  $G$ -action to  $TQ$  induces an equivariant momentum map  $J : TQ \rightarrow \mathfrak{g}^*$  given by

$$J(v_q) \cdot \xi = \mathbb{F}L(v_q) \cdot \xi_Q(q) \quad (1)$$

for  $v_q \in T_q Q$ ,  $\xi \in \mathfrak{g}$ . Let  $\mu \in \mathfrak{g}^*$  be a weakly regular value of  $J$  and assume that  $p_\mu : Q \rightarrow Q/G_\mu$ ,  $TQ \rightarrow (TQ)/G_\mu$  are principal  $G_\mu$ -bundles, where  $G_\mu$  is the coadjoint isotropy group at  $\mu$ . By conservation of  $J$ ,  $G_\mu$  acts on  $J^{-1}(\mu)$  so we can form the reduced space  $(TQ)_\mu = J^{-1}(\mu)/G_\mu$ . Given is  $v_q \in J^{-1}(\mu)$  and an integral curve  $c_\mu(t)$  of the induced vector field  $Z_\mu \in \mathfrak{X}((TQ)_\mu)$ ,  $c_\mu(0) = [v_q]$ ; we want to explicitly find the integral curve  $c(t)$  satisfying  $c(0) = v_q$ . For this purpose we formulate the tangent bundle version of Theorem 2.3 of the previous section.



**3.3 Proposition** Let  $\mu' = \mu|_{\mathfrak{g}_\mu}$  and  $J^\mu : TQ \rightarrow \mathfrak{g}_\mu^*$  be given by  $J^\mu(v_q) \cdot \eta = FL(v) \cdot \eta_Q(q)$  for all  $\eta \in \mathfrak{g}_\mu$ . Assume that there is a vector field  $Y_\mu \in \mathfrak{X}(Q)$  such that  $Y_\mu(Q) \subset (J^\mu)^{-1}(\mu')$  and  $Y_\mu$  is  $G$ -equivariant:  $Y_\mu(g \cdot q) = g \cdot Y_\mu(q)$ . Then there is an embedding  $\phi_\mu : (TQ)_\mu = J^{-1}(\mu)/G_\mu \rightarrow T(Q/G_\mu)$  whose range is a vector subbundle with base  $Q/G_\mu$ . This embedding is onto if and only if  $\mathfrak{g} = \mathfrak{g}_\mu$ . If  $L : TQ \rightarrow \mathbb{R}$  is a  $G$ -invariant Lagrangian whose Lagrangian vector field  $Z \in \mathfrak{X}(TQ)$  is a second order equation, let  $c(t)$  denote an integral curve of  $Z$  and let  $q(t) = \tau(c(t))$  be the corresponding base integral curve, where  $\tau : TQ \rightarrow Q$  is the tangent bundle projection. Let  $q_\mu(t) = \rho_\mu(q(t))$  be the corresponding curve in  $Q/G_\mu$  and  $\tilde{Y}_\mu \in \mathfrak{X}(Q/G_\mu)$  the vector field induced by  $Y_\mu$ . Then the integral curve  $c_\mu(t)$  of  $Z_\mu$  covered by  $c(t)$  is  $\phi_\mu^{-1}(q'_\mu(t) - \tilde{Y}_\mu(q_\mu(t)))$ . The curve  $q_\mu(t)$  is the image of  $c_\mu(t)$  by  $Y_\mu$  followed by the projection  $T(Q/G_\mu) \rightarrow Q/G_\mu$ .

**Proof** The first part of the proof is a restatement of the standard proof of Theorem 2.3. Indeed,  $Y_\mu$  induces, by equivariance, the vector field  $\tilde{Y}_\mu$  on  $Q/G_\mu$ :  $\tilde{Y}_\mu \circ \rho_\mu = T\rho_\mu \circ Y_\mu$ . Define the projection  $\tau_\mu : (TQ)_\mu \rightarrow Q/G_\mu$  by  $\tau_\mu([v_q]) = [q]$ , so that  $\rho_\mu \circ \tau = \tau_\mu \circ \pi_\mu$ , where, as usual,  $\pi_\mu : J^{-1}(\mu) \rightarrow (TQ)_\mu$  is the canonical projection. Let  $t_\mu : (J^\mu)^{-1} \rightarrow (J^\mu)^{-1}(0)$  be given by  $t_\mu(v_q) = v_q - Y_\mu(q)$  and let  $\phi_\mu : (TQ)_\mu \rightarrow T(Q/G_\mu)$  be the induced map,  $\phi_\mu \circ \pi_\mu = T\rho_\mu \circ t_\mu$ , defined on the set  $J^{-1}(\mu)$ . Then  $\phi_\mu$  is an embedding and it is easy to see that it is onto iff  $\mathfrak{g} = \mathfrak{g}_\mu$  by comparing  $(J^\mu)^{-1}(0)$  with  $J^{-1}(0)$ .

For the second part, let  $c_\mu(t) = \pi_\mu(c(t))$  be the integral curve of  $Z_\mu$  covered by  $c(t)$ . Then  $q_\mu(t) = \rho_\mu(q(t)) = (\rho_\mu \circ \tau)(c(t)) = (\tau_\mu \circ \pi_\mu)(c(t)) = \tau_\mu(c_\mu(t))$ . Let us show that  $q'_\mu(t) - \tilde{Y}_\mu(q_\mu(t))$  is in the range of  $\phi_\mu$ . We have

$$\begin{aligned} q'_\mu(t) - \tilde{Y}_\mu(q_\mu(t)) &= T\rho_\mu(q'(t)) - q'_\mu(t) - \tilde{Y}_\mu(\rho_\mu(q(t))) \\ &= T\rho_\mu(q'(t)) - T\rho_\mu(Y_\mu(q(t))) = T\rho_\mu(q'(t) - Y_\mu(q(t))). \end{aligned}$$

Since  $Z$  is a second order equation,  $q'(t) = c(t)$ . By conservation of  $J$ ,  $c(t) \in J^{-1}(\mu)$  for all  $t$ , so  $q'(t) - Y_\mu(q(t)) = t_\mu(q'(t))$  and thus

$$q'_\mu(t) - \tilde{Y}_\mu(q_\mu(t)) = (T\rho_\mu \circ t_\mu)(q'(t)) = (\phi_\mu \circ \pi_\mu)(q'(t)) = (\phi_\mu \circ \pi_\mu)(c(t)) = \phi_\mu(c_\mu(t)).$$

Therefore, the integral curve  $c_\mu(t)$  covered by  $c(t)$  equals  $c_\mu(t) = \phi_\mu^{-1}(q'_\mu(t) - \tilde{Y}_\mu(q_\mu(t)))$  and the proposition is proved. ■

In general,  $Z_\mu$  is *not* a second order equation. However, its integral curves are still uniquely determined by the corresponding base integral curves *i.e.*, by their projections on  $Q/G_\mu$ . We next turn to the question of reconstruction of dynamics. We are given a base integral curve  $q_\mu(t)$  of  $Z_\mu$  determined by the initial conditions  $q_\mu(0)$  and  $q'_\mu(0)$  and an equivariant vector field  $Y_\mu$  with values in  $(J^\mu)^{-1}(\mu')$ , implementing the embedding of  $(TQ)_\mu$  into  $T(Q/G_\mu)$ . We want to reconstruct the integral curve  $c(t) = q'(t)$  of  $Z$  covering  $c_\mu(t) = \varphi_\mu^{-1}(q_\mu(t) - \tilde{Y}_\mu(q_\mu(t)))$ . For this purpose, let  $\gamma \in \Omega^1(Q; \mathfrak{g}_\mu)$  be a connection for the principal  $G_\mu$ -bundle  $\rho_\mu : Q \rightarrow Q/G_\mu$ . Horizontally lift  $q_\mu(t)$  to a curve  $q_h(t)$  in  $Q$  with  $q_h(0) = q(0)$ :

$$\rho_\mu(q_h(t)) = q_\mu(t) \text{ and } \gamma(q_h(t)) \cdot q'_h(t) = 0. \quad (2)$$

Define the induced connection  $\gamma^\mu \in \Omega^1(J^{-1}(\mu); \mathfrak{g}_\mu)$  by

$$\gamma^\mu(V_{v_q}) = \gamma(q) \cdot T\tau(V_{v_q}) \quad (3)$$

for  $v_q \in T_q Q$  and  $V_{v_q} \in T_{v_q}(TQ)$ . (Note that if  $L$  is hyperregular, *i.e.*,  $\mathbb{F}L : TQ \rightarrow T^*Q$  is a diffeomorphism, then  $(\mathbb{F}L)^* \tilde{\gamma}^\mu = \gamma^\mu$ , for  $\tilde{\gamma}^\mu$  the connection given in Corollary 2.4). In Step i of the reconstruction procedure we horizontally lift  $c_\mu(t)$  to a curve  $d(t)$  in  $J^{-1}(\mu)$  with  $d(0) = c(0)$ :  $\pi_\mu(d(t)) = c_\mu(t)$  and  $\gamma^\mu(d(t)) \cdot d'(t) = 0$ . To determine  $d(t)$ , we begin by showing that  $\tau(d(t)) = q_h(t)$ . Indeed,  $\tau(d(t))$  covers  $q_\mu(t)$ :  $\rho_\mu(\tau(d(t))) = (\tau_\mu \circ \pi_\mu)(d(t)) = \pi_\mu(c_\mu(t)) = q_\mu(t)$ . Moreover, since  $(\tau \circ d)'(t) = T\tau(q'(t))$ , the horizontality condition on  $d(t)$  becomes  $\gamma(\tau(d(t))) \cdot (\tau \circ d)'(t) = 0$ , *i.e.*,  $\tau(d(t))$  is horizontal. Finally, since  $\tau(d(0)) = \tau(c(0)) = q(0) = q_h(0)$ , the equality  $\tau(d(t)) = q_h(t)$  follows by uniqueness of horizontal lift.

Next, we show that  $q'_h(t)$  is the  $\gamma$ -horizontal part of  $d(t)$ . Indeed, since  $\pi_\mu(d(t)) = c_\mu(t) = \varphi_\mu^{-1}(q'_\mu(t) - \tilde{Y}_\mu(q_\mu(t)))$  by Proposition 3.3, we have

$$\begin{aligned} q'_\mu(t) - \tilde{Y}_\mu(q_\mu(t)) &= (\varphi_\mu \circ \pi_\mu)(c(t)) = (\varphi_\mu \circ \pi_\mu)(d(t)) = (T\rho_\mu \circ t_\mu)(d(t)) \\ &= T\rho_\mu(d(t) - Y_\mu(q_h(t))) = T\rho_\mu(d(t)) - \tilde{Y}_\mu(\rho_\mu(q_h(t))) \\ &= T\rho_\mu(d(t)) - \tilde{Y}_\mu(q_\mu(t)), \end{aligned}$$

whence

$$q'_\mu(t) = T\rho_\mu(d(t)).$$

Since  $T\rho_\mu(q'_h(t)) = q'_\mu(t)$ , it follows that  $d(t) - q'_h(t)$  is vertical, so  $q'_h(t)$  is the horizontal part of  $d(t)$  and thus there is a unique  $\xi(t) \in \mathfrak{g}_\mu$  such that

$$d(t) = q'_h(t) + \xi(t)_Q(q_h(t)). \quad (4)$$

Note that  $\xi(0)_Q(q)$  is the vertical part of the initial condition  $v_q$ . The only remaining condition on  $d(t)$  is that it lie in  $J^{-1}(\mu)$ , i.e., that

$$\mathbb{F}L(q'_h(t) + \xi(t)_Q(q_h(t))) \cdot \eta_Q(q_h(t)) = \mu \cdot \eta \quad (5)$$

for all  $\eta \in \mathfrak{g}$ . This condition uniquely determines  $\xi(t)$  by the formula  $\xi(t) = \gamma(q_h(t)) \cdot d(t)$ . We have proved the following:

**3.4 Corollary** *Under the hypotheses and notations of Proposition 3.3, let  $q_\mu(t)$  be the base integral curve of  $Z_\mu \in \mathfrak{X}((TQ)_\mu)$  with initial conditions  $q_\mu(0)$ ,  $q'_\mu(0)$  and  $c_\mu(t) = \varphi_\mu^{-1}(q'_\mu(t) - \tilde{Y}_\mu(q_\mu(t)))$  the corresponding integral curve of  $Z_\mu$ . Let  $\gamma \in \Omega^1(Q; \mathfrak{g}_\mu)$  be a connection on the principal  $G_\mu$ -bundle  $\rho_\mu: Q \rightarrow Q/G_\mu$ . Then the integral curve  $c(t) = q'(t)$  of  $Z \in \mathfrak{X}(TQ)$  covering  $c_\mu(t)$  with initial condition  $v_q \in \pi_\mu^{-1}(c_\mu(0))$  is found in the following way:*

- i horizontally lift  $q_\mu(t)$  to a curve  $q_h(t)$  in  $Q$ ,  $q_h(0) = q$ ;
- ii determine  $\xi(t) \in \mathfrak{g}_\mu$  from the system

$$\mathbb{F}L(q'_h(t) + \xi(t)_Q(q_h(t))) \cdot \eta_Q(q_h(t)) = \mu \cdot \eta$$

for all  $\eta \in \mathfrak{g}$ ; this implies that the horizontal part of the initial condition  $v_q$  is  $q'_h(0)$  and the vertical part is  $\xi(0)_Q(q)$ ;

- iii solve the non-autonomous ordinary differential equation  $g'(t) = T_e L_{g(t)} \xi(t)$  with initial condition  $g(0) = e$  on the Lie subgroup  $G_\mu$ .

Then the base integral curve  $q(t)$  of  $Z$  with initial conditions  $q(0) = q$ ,  $q'(0) = v_q$  is given by  $q(t) = g(t) \cdot q_h(t)$  and the integral curve of  $Z$  with initial condition  $v_q$  is  $q'(t) = g(t) \cdot (q'_h(t) + \xi(t)_Q(q_h(t)))$ .

The vector field  $Y_\mu \in \mathfrak{X}(Q)$  postulated in the hypotheses of Proposition 3.3 can be chosen consistent with the connection  $\gamma^\mu$  given by (3). Namely, define

$$Y_\mu(g) = \mathbb{F}L(\mu' \cdot \gamma(g)), \quad (6)$$

i.e.,  $Y_\mu(g)$  is the Legendre transform of the  $\mu'$ -component of the connection  $\gamma \in \Omega^1(Q; \mathfrak{g}_\mu)$ . This choice for hyperregular Lagrangians (i.e., if  $\mathbb{F}L$  is a vector bundle isomorphism between  $TQ$  and  $T^*Q$ ) allows one to pass freely between the Lagrangian and Hamiltonian point of views both at the unreduced and reduced levels. We shall use this remark in §4D.

### §3D Simple Mechanical Systems

To specialize this corollary to the case of Lagrangians of kinetic minus potential type, let  $(Q, \langle \cdot, \cdot \rangle)$  be a Riemannian manifold with positive definite,  $G$ -invariant metric  $\langle \cdot, \cdot \rangle$  and let  $V : Q \rightarrow \mathbb{R}$  be a  $G$ -invariant potential. Then the classical Lagrangian  $L(v_q) = \frac{1}{2} \|v_q\|^2 - V(q)$  is  $G$ -invariant. The metric induces a principal connection on  $\rho_\mu : Q \rightarrow Q/G_\mu$  by declaring the horizontal subbundle to equal the orthogonal complement of the vertical subbundle. The condition determining  $\xi(t) \in \mathfrak{g}_\mu$  becomes

$$\langle q'_h(t) + \xi(t)_Q(q_h(t)), \eta_Q(q_h(t)) \rangle = \mu \cdot \eta \quad (1)$$

for all  $\eta \in \mathfrak{g}$ . Condition (1) implies that  $q'_h(0)$  is the horizontal part and  $\xi(0)_Q(q)$  the vertical part of the initial condition  $v_q \in T_q Q$ . Split  $\mathfrak{g} = \mathfrak{g}_\mu \oplus E$  for some complement  $E$  of  $\mathfrak{g}_\mu$ . If in (1)  $\eta$  is taken to lie in  $\mathfrak{g}_\mu$ , then by the definition of the connection,

$$\langle \xi(t)_Q(q_h(t)), \eta_Q(q_h(t)) \rangle = \mu \cdot \eta \quad (2)$$

for all  $\eta \in \mathfrak{g}_\mu$ , an equality uniquely determining  $\xi(t)_Q(q_h(t))$  for every  $t$ , and hence by freeness of the  $G_\mu$ -action, uniquely determining  $\xi(t) \in \mathfrak{g}_\mu$ . There are still  $\dim G - \dim G_\mu$  equations to be satisfied; these hold automatically by the previous corollary. We have thus proved the following:

**3.5 Corollary** *In the hypotheses of Proposition 3.3, let  $L(v_q) = \frac{1}{2} \|v_q\|^2 - V(q)$ , where  $\langle \cdot, \cdot \rangle$  is a  $G$ -invariant positive definite Riemannian metric on  $Q$  and  $V : Q \rightarrow \mathbb{R}$  is  $G$ -invariant. Let  $Z \in \mathfrak{X}(TQ)$  be the Lagrangian vector field of  $L$  and let  $Z_\mu \in \mathfrak{X}((TQ)_\mu)$  be the induced vector field on the reduced space. Let  $v_q \in J^{-1}(\mu)$  and let  $q_\mu(t)$  be the base integral curve of  $Z_\mu$  with initial conditions  $\pi_\mu(v_q)$ . Then the integral curve  $c(t)$  of  $Z$  with initial condition  $c(0) = v_q$  is found in the following way:*

- i endow the principal  $G_\mu$ -bundle  $\rho_\mu : Q \rightarrow Q/G_\mu$  with the connection  $\gamma_{\text{mech}}$  whose horizontal subbundle is the orthogonal complement relative to  $\langle \cdot, \cdot \rangle$  of the vertical subbundle;
- ii horizontally lift  $q_\mu(t)$  to a curve  $q_h(t)$  in  $Q$  satisfying  $q_h(0) = q$ ;
- iii determine  $\xi(t) \in \mathfrak{g}_\mu$  from the algebraic system  $\langle \xi(t)_Q(q_h(t)), \eta_Q(q_h(t)) \rangle = \mu \cdot \eta$  for all  $\eta \in \mathfrak{g}_\mu$ ; this implies that  $q'_h(0)$  and  $\xi(0)_Q(q)$  are the horizontal and vertical parts of the initial condition  $v_q$ ;
- iv solve the non-autonomous equation  $g'(t) = T_e L_{g(t)} \xi(t)$ ,  $g(0) = e$  in  $G_\mu$ .

Then  $q(t) = g(t) \cdot q_h(t)$  is the base integral curve of  $Z$  with initial conditions  $q(0) = q$ ,  $q'(0) = v_q$  and  $c(t) = q'(t) = g(t) \cdot (q'_h(t) + \xi(t)_Q(q_h(t)))$  is the integral curve of  $Z$  with initial condition  $c(0) = v_q$ .

It is easy to check that the Legendre transform of  $c(t)$  gives the integral curve of the associated Hamiltonian system as obtained in 2.1 relative to the connection in 2.6.

There are several important situations when step **iv** can be carried out explicitly.

**a** If  $G_\mu$  is abelian, the equation  $g'(t) = T_e L_{g(t)} \xi(t)$ ,  $g(0) = e$ , has solution given by  $g(t) = \exp \int_0^t \xi(s) ds$ . If  $G_\mu = S^1$ ,  $\xi(s)$  can be explicitly determined from condition **iii** and hence the reconstruction method of Corollary 3.5 has an explicit solution. Indeed, if we denote by  $\zeta \in \mathfrak{g}_\mu$  a generator of  $\mathfrak{g}_\mu$ , then  $(a(t)\zeta)_Q = a(t)\zeta_Q$ , so that by **ii** we get  $\mu \cdot \zeta = \langle a(t)\zeta_Q(q_h(t)), \zeta_Q(q_h(t)) \rangle = a(t) \|\zeta_Q(q_h(t))\|^2$  and hence

$$a(t) = \frac{\mu \cdot \zeta}{\|\zeta_Q(q_h(t))\|^2}.$$

Thus writing  $g(t) = \exp(\theta(t)\zeta)$ , the solution of the equation in **iv** is given by

$$\theta(t) = (\mu \cdot \zeta) \int_0^t \frac{ds}{\|\zeta_Q(q_h(s))\|^2}.$$

**3.6 Corollary** Let  $(Q, \langle \cdot, \cdot \rangle)$  be a Riemannian manifold with a  $G$ -invariant metric and  $V : Q \rightarrow \mathbb{R}$  be a  $G$ -invariant function. Assume  $G_\mu$  equals  $S^1$  or  $\mathbb{R}$  and let  $\zeta$  be a generator of  $\mathfrak{g}_\mu$ . If  $v_q \in J^{-1}(\mu) = \{u \in TQ \mid \langle u, \zeta_Q(q) \rangle = \mu \cdot \zeta\}$  and  $q_\mu$  is the base integral curve of the induced vector field  $Z_\mu \in \mathfrak{X}((TQ)_\mu)$  with initial conditions  $\pi_\mu(v_q)$ , the base integral curve and solution of the Lagrangian vector field  $Z$  given by  $L(v_q) = \frac{1}{2} \|v_q\|^2 - V(q)$  with initial condition  $v_q$  is given by

$$q(t) = \exp(\theta(t)\zeta) \cdot q_h(t) \text{ and } c(t) = q'(t) = \exp(\theta(t)\zeta) \cdot \left( q'_h(t) + \frac{\mu \cdot \zeta}{\|\zeta_Q(q_h(t))\|^2} \zeta_Q(q_h(t)) \right), \quad (4)$$

where

$$\theta(t) = (\mu \cdot \zeta) \int_0^t \frac{ds}{\|\zeta_Q(q_h(s))\|^2} \quad (5)$$

and  $q_h(t)$  is the horizontal lift of  $q_\mu(t)$  satisfying  $q_h(0) = q$ .

**b** A second case in which the equation  $g'(t) = T_e L_{g(t)} \xi(t)$  can be solved, even if  $G_\mu$  is non-abelian when  $\xi(t) = \xi(0) = \xi$  is constant. Then  $g(t) = \exp(t\xi)$  is the solution. This holds if  $g$  admits a non-degenerate metric  $\langle \cdot, \cdot \rangle$  satisfying

$$\langle \xi, \eta \rangle = \langle \xi_Q(q), \eta_Q(q) \rangle \quad (6)$$

for any  $q \in Q$ . This condition is prohibitively strong. For example, it implies that  $\langle \cdot, \cdot \rangle$  defines a positive definite metric on  $g$ . Denoting by  $\Phi_g : Q \rightarrow Q$  the action of  $G$  on  $Q$ , this relation implies by  $G$ -invariance of the metric  $\langle \cdot, \cdot \rangle$  on  $Q$  that:

$$\begin{aligned} \langle \text{Ad}_g \xi, \text{Ad}_g \eta \rangle &= \langle (\text{Ad}_g \xi)_Q(q), (\text{Ad}_g \eta)_Q(q) \rangle = \langle \Phi_g^* \xi_Q(q), \Phi_g^* \eta_Q(q) \rangle \\ &= \langle T\Phi_g(\xi_Q(g^{-1} \cdot q)), T\Phi_g(\eta_Q(g^{-1} \cdot q)) \rangle = \langle \xi_Q(g^{-1} \cdot q), \eta_Q(g^{-1} \cdot q) \rangle = \langle \xi, \eta \rangle, \end{aligned}$$

i.e.,  $\langle \cdot, \cdot \rangle$  must in addition be bi-invariant. This excludes semisimple Lie algebras of non-compact type. What is even worse, if  $G$  is compact and we are interested in classical Lagrangian systems on  $TG$  defined by left-invariant metrics on  $G$ , this condition forces the metric  $\langle \cdot, \cdot \rangle$  to be bi-invariant:  $\langle \xi, \eta \rangle = \langle \xi_G(e), \eta_G(e) \rangle = \langle \xi, \eta \rangle$ . This hypothesis thus fails for the free rigid body (unless it is spherically symmetric). A class of compact Lie group actions where such a condition holds occurs in Kaluza-Klein theories; see Montgomery [1989]. Using condition iii in Corollary 3.5, condition (6) implies that  $\xi(t)$  is constant:  $\langle \xi(t), \eta \rangle = \langle \xi(t)_Q(q_h(t)), \eta_Q(q_h(t)) \rangle = \mu \cdot \eta$  and therefore  $\xi(t) = \xi$ , where  $\langle \xi, \cdot \rangle = \mu | g_\mu$ .

**3.7 Corollary** Let  $(Q, \langle \cdot, \cdot \rangle)$  be a Riemannian manifold with a  $G$ -invariant metric and a  $G$ -invariant Lagrangian.  $L = \frac{1}{2} \|v_q\|^2 - V(q)$ . Assume  $g$  carries a positive definite bi-invariant metric  $\langle \cdot, \cdot \rangle$  satisfying  $\langle \xi, \eta \rangle = \langle \xi_Q(q), \eta_Q(q) \rangle$  for all  $q \in Q$ . Then the solution of the Lagrangian vector field on  $TQ$  with initial condition  $v_q \in J^{-1}(\mu)$  is determined from the solution of the reduced system by following steps i - iv in Corollary 3.5 with  $g(t) = \exp(t\xi)$  for  $\xi \in g_\mu$  determined by  $\langle \xi, \cdot \rangle = \mu | g_\mu$ .

**c** Another class of problems where  $g'(t) = T_e L_{g(t)} \xi(t)$  is explicitly solvable is as follows. We begin by trying to find a real valued function  $f(t)$  such that  $g(t) = \exp(f(t)\xi(t))$  solves this equation. Since  $g(0) = e$ , we require  $f(0) = 0$ . We have

$$g'(t) = (T_{f(t)\xi(t)} \exp)(f'(t) \xi(t) + f(t) \xi'(t)).$$

From the formula  $(T_\eta \exp) \cdot \eta = T_e L_{\exp \eta} \cdot \eta$ , we get

$$T_e L_{\exp f(t)\xi(t)} \xi(t) = \frac{1}{f(t)} T_e L_{\exp f(t)\xi(t)} (f(t) \xi(t))$$

$$= \frac{1}{f(t)} (T_{f(t)\xi(t)} \exp)(f(t) \xi(t)) = (T_{f(t)\xi(t)} \exp)(\xi(t))$$

so that  $g'(t) = T_e L_{g(t)} \xi(t)$  for  $g(t) = \exp f(t) \xi(t)$  is satisfied if and only if

$$f'(t) \xi(t) + f(t) \xi'(t) = \xi(t).$$

This requires  $\xi'(t) = \alpha(t) \xi(t)$  in which case  $f(t)$  is given by

$$f(t) = \int_0^t \exp \left[ \int_t^s \alpha(r) dr \right] ds.$$

This procedure can be generalized as follows. Consider  $f(t) = \sum_{i=1}^N f_i(t) \xi^{(i)}(t)$  and assume

i that it commutes with  $\xi(t)$  for all  $t$ , and

ii  $\xi^{(N+1)}(t) = \sum_{i=1}^N \alpha_i(t) \xi^{(i)}(t)$ , for some functions  $\alpha_0(t), \dots, \alpha_N(t)$ .

Then proceeding as before and using the formula  $(T_\eta \exp)(\xi) = T_e L_{\exp \eta}(\xi)$  for  $[\xi, \eta] = 0$  one is led to the system of ordinary differential equations with variable coefficients  $f' = Af + v$ , where

$$f = \begin{bmatrix} f_0 \\ \cdot \\ \cdot \\ f_N \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & \dots & 0 & -\alpha_0(t) \\ -1 & 0 & \dots & 0 & -\alpha_1(t) \\ 0 & -1 & \dots & 0 & -\alpha_2(t) \\ 0 & 0 & \dots & -1 & -\alpha_N(t) \end{bmatrix}.$$

Here  $g = \exp(f_1 \xi^1 + \dots + f_N \xi^N)$  is sometimes called the *Magnus expansion*.

d Another method by which one can treat  $g' = T_e L_g \xi$  was pointed out by P S. Krishnaprasad. If  $G_\mu$  is solvable, this method leads to explicit solutions. Write

$$g = \exp(f_1 x_1) \exp(f_2 x_2) \dots \exp(f_n x_n)$$

for a suitable basis  $\{x_i\}$  of  $\mathfrak{g}_\mu$ . For solvable  $G_\mu$ , Wei and Norman [1984] showed that the  $f_i$ 's can be obtained by quadrature. For  $\mathfrak{so}(3)$  it leads to some nonlinear equations for  $f_1, f_2, f_3$  which do not seem to be integrable by quadratures.

## §4 Ehresmann Connections and Holonomy

In this section, we review some of the relevant facts from the theory of connections that will be needed. The exposition is by no means complete and is provided for the convenience of the reader. For applications to phases, the context of principal  $G$ -bundles is not adequate for all the examples, so we work in the larger framework of Ehresmann connections. These connections were introduced in Ehresman [1950]. Since our needs are modest, we do not attempt to survey the vast modern literature on connections, but we do make some historical remarks below.

### §4A Ehresmann Connections

Let  $\pi : E \rightarrow M$  be a surjective submersion,  $V = \ker T\pi$  the *vertical subbundle* of  $TE$  and  $\mathfrak{X}_{\text{vert}}(E; M)$  its space of sections, elements of which are called *vertical vector fields*. An *Ehresmann connection* on  $\pi : E \rightarrow M$  is a smooth subbundle  $H$  of  $TE$  called the *horizontal subbundle* such that  $H \oplus V = TE$ . The space of sections of  $H$ , denoted by  $\mathfrak{X}_{\text{hor}}(E; M)$  is the space of *horizontal vector fields*. Since  $T\pi|_H : H \rightarrow TM$  is an isomorphism on every fiber it has a fiberwise inverse called the *horizontal lift operator*  $\text{hor}_p : T_{\pi(p)}M \rightarrow T_pE$  for all  $p \in E$ , i.e.,  $\text{hor}_p = (T_p\pi|_{H_p})^{-1}$ . Since  $H$  is a smooth subbundle of  $TE$ , the horizontal lift defines a linear map  $\text{hor} : \mathfrak{X}(M) \rightarrow \mathfrak{X}_{\text{hor}}(E; M)$  by  $(\text{hor } X)(p) = \text{hor}_p(X(p))$ .

In general, a *lift* of a vector  $v \in T_mM$  is a vector field  $\text{lift } v$  along (and in general not tangent to)  $\pi^{-1}(m)$  such that  $T_m\pi(\text{lift } v) = v$  and  $v \mapsto \text{lift } v$  is linear. Given  $X \in \mathfrak{X}(M)$ , define  $\text{lift } X : E \rightarrow TE$  by  $(\text{lift } X)(p) = (\text{lift } X)(\pi(p))(p)$  and note that  $(\text{lift } X)(p) \in T_pE$ . We say that the lift is *smooth* if  $\text{lift } X \in \mathfrak{X}(E)$  for all  $X \in \mathfrak{X}(M)$ . With these definitions, we see that an Ehresmann connection is alternatively defined by a smooth lift  $\text{hor} : \mathfrak{X}(M) \rightarrow \mathfrak{X}(E)$  such that if  $H_p = \text{hor}_p(T_{\pi(p)}M)$ , then  $H = \bigcup_{p \in E} H_p$  is a smooth subbundle of  $TE$  complementary to  $V$ .

For any  $u \in T_pE$ , let  $u = u^{\text{hor}} + u^{\text{vert}}$  be its decomposition into its horizontal and vertical parts. The vertical projection  $\gamma(p) : T_pE \rightarrow V_p$  given by  $\gamma(p)(u) = u^{\text{vert}}$  defines a smooth  $V$ -valued one-form  $\gamma \in \Omega^1(E; V)$ , called the *connection one form*. This form satisfies  $\gamma_p = \text{identity on } V_p$ . Conversely, given  $\gamma$  with  $\gamma_p = \text{identity on } V_p$ , it uniquely determines a smooth horizontal lift by

$$\text{hor}_p v = \tilde{v} - \gamma(p)(\tilde{v}) \quad (1)$$



where  $v \in T_{\pi(p)}M$  and  $\tilde{v} \in T_pE$  is an arbitrary vector satisfying  $T_e\pi(\tilde{v}) = v$ . Thus an *Ehresmann connection* is equivalently described by a horizontal subbundle, a horizontal lift, or a vector bundle valued connection one-form which is the identity on the vertical subbundle.

If  $\alpha \in \Omega^k(E)$ , its *covariant exterior derivative*  $D\alpha \in \Omega^{k+1}(E)$  is defined by

$$D\alpha(X_0, \dots, X_k) = d\alpha(X_0^{\text{hor}}, \dots, X_k^{\text{hor}}), \quad (2)$$

where  $X_0, \dots, X_k \in \mathfrak{X}(E)$  and  $X_i^{\text{hor}}$  is the horizontal part of  $X$ , taken pointwise. In terms of pointwise operations, we can write, with a minor abuse of notation,

$$(D\alpha(m)(v_0, \dots, v_k))(p) = D\alpha(p)(\text{hor}_p v_0, \dots, \text{hor}_p v_k), \quad \text{for } m = \pi(p). \quad (3)$$

If  $\lambda \in \Omega^k(E; V)$  is a  $V$ -valued  $k$ -form, where  $V$  is the vertical subbundle of  $TE$ , the *covariant derivative*  $D\lambda \in \Omega^{k+1}(E; V)$  is defined using *Cartan's formula*:

$$\begin{aligned} D\lambda(X_0, \dots, X_k) = & \sum_{i=0}^k (-1)^i [X_i^{\text{hor}}, \lambda(X_0^{\text{hor}}, \dots, \check{X}_i, \dots, X_k^{\text{hor}})]^{\text{vert}} \\ & + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \lambda([X_i^{\text{hor}}, X_j^{\text{hor}}], X_0^{\text{hor}}, \dots, \check{X}_i, \dots, \check{X}_j, \dots, X_k^{\text{hor}}) \end{aligned} \quad (4)$$

where  $X_0, \dots, X_k \in \mathfrak{X}(E)$  and  $\check{\phantom{x}}$  above a vector field means that it is deleted. As above, the covariant derivative of  $\lambda$  can be thought of as a  $(k+1)$ -form on  $M$  with values in  $\mathfrak{X}_{\text{vert}}(E; M)$ , or equivalently, as a  $(k+1)$ -linear skew symmetric map  $\tilde{D}\lambda : \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \rightarrow \mathfrak{X}_{\text{vert}}(E; M)$ , namely

$$\tilde{D}\lambda(Y_0, \dots, Y_k) = D\lambda(\text{hor } Y_0, \dots, \text{hor } Y_k). \quad (5)$$

If  $\gamma \in \Omega^1(E; V)$  is a connection one-form, then  $D\gamma = \Omega \in \Omega^2(E; V)$  is called the *curvature* of  $\gamma$ . Sometimes we shall write  $\text{curv}(\gamma)$  for the curvature of  $\gamma$ . Since  $\gamma$  annihilates horizontal vectors, (4) gives

$$\Omega(X, Y)(p) = -\gamma(p)([X^{\text{hor}}, Y^{\text{hor}}](p)) \quad (6)$$

and its induced map on the base  $\bar{\Omega} : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}_{\text{vert}}(E; M)$  is given by  $\bar{\Omega}(U, W) = \Omega(\text{hor } U, \text{hor } W) = \text{hor}[U, W] - [\text{hor } U, \text{hor } W]$ ; this follows from the identity  $[\text{hor } U, \text{hor } W]^{\text{hor}} = \text{hor}[U, W]$  which we now prove. By definition of the horizontal lift,  $[\text{hor } U, \text{hor } W]^{\text{hor}}$  is horizontal and  $T\pi([\text{hor } U, \text{hor } W]^{\text{hor}}) = T\pi([\text{hor } U, \text{hor } W]) = [U, W]$  because  $\text{hor } U$  and  $U$

are  $\pi$ -related and  $\pi$ -relatedness is bracket preserving. Note that  $\Omega$  and  $\bar{\Omega}$  determine each other uniquely. We just saw how  $\Omega$  induces  $\bar{\Omega}$ . Conversely, let  $\bar{\Omega}$  be given by (6). This determines  $\Omega$  uniquely on horizontal vectors and so declaring  $i_Z\Omega = 0$  for all  $Z \in \mathfrak{X}_{\text{vert}}(E; M)$  determines  $\Omega$  on  $\mathfrak{X}(E) \times \mathfrak{X}(E)$ . The formula for  $\Omega$  follows from the definition of  $\bar{\Omega}$  and the fact that  $\gamma(p)$  is the vertical projection  $T_p E \rightarrow V_p$ .

**Bianchi's identity** states that  $D\Omega = 0$ . This is proved by using the definition of the covariant derivative and the remark that *the Lie bracket of any vertical with any projectable vector field is again vertical*; a vector field on  $E$  is **projectable** if it is  $\pi$ -related to some vector field on  $M$ . For example, the bracket of any horizontally lifted vector field with a vertical vector field is again vertical.

Let  $m(t)$ ,  $t \in [0, 1]$  be a smooth path in  $M$ . A **horizontal lift** of  $m(t)$  is a smooth path  $p(t)$  in  $E$  such that  $\pi(p(t)) = m(t)$  and the tangent vector  $\dot{p}(t)$  to  $p(t)$  is horizontal for every  $t \in [0, 1]$ . If  $\pi: E \rightarrow M$  is a locally trivial fiber bundle, then given a smooth path  $m(t)$ , in  $M$  with  $m_0 = m(0)$ ,  $0 \leq t \leq 1$  and  $p_0 \in \pi^{-1}(m(0))$ , there is a unique locally defined horizontal lift  $p(t)$  of  $m(t)$  satisfying  $p(0) = p_0$ . If  $p(t)$  can be extended for all  $t \in [0, 1]$ , we call the connection **complete**. We will generally assume this is the case; if  $E$  is a locally trivial finite dimensional fiber bundle with structure group acting transitively on the fibers and if the Ehresman connection is invariant under the action, then the connection is complete using the following local existence argument, compactness of  $[0, 1]$  and the fact that the time of existence is independent of the position on a given fiber. In particular, completeness holds for principal connections. To prove local existence and uniqueness of horizontal lifts, it suffices to do so for the case of a trivial bundle  $E = M \times F$  with  $\pi: M \times F \rightarrow M$  the projection. The equality

$$\gamma(m, f)(u, v) = v + \lambda(m, f)(u) \quad (7)$$

for  $m \in M$ ,  $f \in F$ ,  $u \in T_m M$ , and  $v \in T_f F$  defines a bijective correspondence between connection one-forms  $\gamma \in \Omega^1(M \times F; V)$  and smooth sections of the bundle  $L(TM, TF) \rightarrow M \times F$  of vector bundle maps from  $TM$  to  $TF$  with base  $M \times F$ . Thus  $(u, v)$  is horizontal iff  $v = -\lambda(m, f)(u)$ , so that if  $m(t)$  is a path in  $M$ , define  $p(t) = (m(t), f(t))$  with  $f(t)$  the solution of the time dependent differential equation

$$\frac{df(t)}{dt} = -\lambda(m(t), f(t))\dot{m}(t) \quad (8)$$

with initial condition  $f(0) = f_0$ , where  $p_0 = (m_0, f_0)$ . By local existence and uniqueness for differential equations this defines  $f(t)$  (and hence  $p(t)$ ) for small  $t$ . If the path  $m(t)$  is constant and equal to  $m_0$ , then  $p(t)$  is also constant and equal to  $p_0$ . Let  $m(t)$  be a general path,  $m(0) =$

$m_0$  and  $m(1) = m_1$ . If  $e_p(t)$  is the horizontal lift of  $m(t)$  satisfying  $e_p(0) = p \in \pi^{-1}(m_0)$ , then the map sending  $p$  to  $e_p(s)$  defines, by uniqueness of horizontal lift, a bijection from  $\pi^{-1}(m_0)$  to  $\pi^{-1}(m_1)$ . By smooth dependence of solutions of ordinary differential equations on initial conditions, it follows that this map is a diffeomorphism; it is called the *parallel transport operator*.

### Historical Remarks

We thank Arthur Fischer for the following remarks. An Ehresmann connection  $\gamma$  on a fiber bundle  $\pi: E \rightarrow M$  induces an Ehresmann connection  $T\gamma$  on the tangent bundle  $T\pi: TE \rightarrow TM$  in a functorial way. (This is due to Kobayashi [1957] for the principal bundle case and to Vilms [1967] for the vector bundle case). More interestingly, the connection  $\gamma$  also induces a connection  $\gamma^B$  on the vertical bundle  $v_E: VE \rightarrow E$ , called the *Berwald connection* (see Yano and Okubo [1961], Vilms [1968], and Tong Van Duc [1975].) The original work of Berwald [1926, 1933, 1939] was for the case of  $E = TM$ . In particular, formula (4) above may be viewed as the covariant exterior derivative with respect to the Berwald connection  $\gamma^B$  induced by the given Ehresmann connection. ♦

### §4B Holonomy

Now let  $m(t)$ ,  $t \in [0, 1]$  be a closed path in  $N$ ,  $m(0) = m_0$ . The diffeomorphism of  $\pi^{-1}(m_0)$  onto itself given by parallel transport along  $m(t)$  is called the *holonomy of the path*  $m(t)$ . It is easy to see that parallel transport sends juxtaposition of loops based at  $m_0$  into the composition of diffeomorphisms of  $\pi^{-1}(m_0)$ . Thus the holonomy operation is a group homomorphism of  $\mathcal{L}(m_0)$ , the loop group at  $m_0$ , to the diffeomorphism group of  $\pi^{-1}(m_0)$ ; its image  $\mathcal{H}(m_0)$  is called the *holonomy group at*  $m_0$ . It is straightforward to see that if  $M$  is connected all holonomy groups are conjugate:  $\mathcal{H}(m_0)$  and  $\mathcal{H}(m_1)$  are conjugate by the parallel transport along any path connecting  $m_0$  to  $m_1$ . Thus if  $M$  is connected, we speak of  $\mathcal{H}$ , the *holonomy group of the connection*.

If  $\Phi: E \times G \rightarrow E$  is a (by tradition, right) Lie group action of  $G$  on  $E$  and the connection  $\gamma$  is  $G$ -invariant, i.e., if  $H_{p \cdot g} = T_p \Phi_g(H_p)$  for all  $p \in E$ ,  $g \in G$ , then parallel transport is equivariant. This happens if  $\pi: E \rightarrow M$  is a principal  $G$ -bundle and  $\gamma \in \Omega^1(E; \mathfrak{g})$  is a principal connection: horizontal lift commutes with the  $G$ -action. Here we have identified the vertical bundle  $V$  with  $M \times \mathfrak{g}$ , which is done by using the group action. This map is  $G$ -equivariant when we take the action on  $E \times \mathfrak{g}$  to be the diagonal one:

$$(e, \xi) \cdot g = (e \cdot g, \text{Ad}_{g^{-1}} \xi).$$

Consequently, the  $G$ -equivariance of  $\lambda$  as a  $\mathfrak{g}$ -valued one-form means that

$$g^*\gamma = \text{Ad}_g \circ \gamma.$$

The holonomy can be realized more explicitly in the principal bundle case by fixing a point  $p \in \pi^{-1}(m_0)$ . Associate to every holonomy operator  $\varphi \in \mathcal{H}(m_0)$  the group element  $g$  of  $G$  by  $\varphi(p) = p \cdot g$ . We will call this the holonomy *measured from*  $p$ . If we measured the holonomy from  $p' = p \cdot g_1$  instead, we would find that  $g$  gets replaced by  $g_1^{-1}gg_1$ . This is because  $\varphi$  commutes with right multiplication.

In what follows, we consider the case of principal bundles for which the computation of the holonomy of a path is theoretically very simple.

**4.1 Proposition** *Let  $\pi : E \rightarrow M$  be a principal  $G$ -bundle and  $\gamma \in \Omega^1(E; \mathfrak{g})$  a principal connection. Let  $c$  be a closed path in  $M$  which is contained in the open set  $U$ . Suppose  $s : U \subset M \rightarrow E$  is a local section and set  $a = s^*\gamma$  so that  $a$  is a  $\mathfrak{g}$ -valued one-form on  $U$ . Let  $g(t)$  be the solution to*

$$\frac{dg(t)}{dt} = -a\left(\frac{dc}{dt}\right) \cdot g(t)$$

with  $g(0) = 1$ . Then the holonomy of  $c$ , measured from  $s(c(0))$  is  $g(1)$ .

**Remark** The solution  $g$  to this equation is written  $\mathcal{P}\exp\left(-\int_c a\right)$  in the physics literature, where  $\mathcal{P}$  denotes "path ordering".

**Proof** The section  $s$  induces a local trivialization of  $E$  in the usual way: write  $e = s(x) \cdot g$ . Let  $(c(t), g(t))$  be the "coordinates" of the horizontal lift  $\bar{c}$  of  $c$  in this trivialization. By the transformation law for connections,

$$\gamma(x, g) = g^{-1}(a + d)g = g^{-1}ag + g^{-1}dg$$

in this trivialization. Here we have used matrix notation for simplicity: that is, we write  $g^{-1}ag = \text{Ad}_{g^{-1}} \circ a$ . The condition that a curve  $\bar{c}$  be horizontal is that  $\bar{c}^*\gamma = 0$  i.e.,  $\gamma\left(\frac{d\bar{c}}{dt}\right) = 0$ . In our coordinates this reads

$$g^{-1}\left(a\left(\frac{dc}{dt}\right) \cdot g + \frac{dg}{dt}\right) = 0 \quad \text{i.e.,} \quad \frac{dg}{dt} = -a\left(\frac{dc}{dt}\right) \cdot g(t). \quad \blacksquare$$

Now assume that  $G$  is abelian; then it is a cylinder  $\mathbb{R}^k \times \mathbb{T}^{s-k}$  and in particular each  $g$  is of the form  $\exp \eta$ . Therefore  $TL_{g^{-1}} \dot{g} = \dot{\eta}$  so, by the proposition, the holonomy element is

$$g(1) = \exp \eta(1) = \exp \left( - \int_0^1 (s^* \gamma)(m(\lambda)) \cdot m'(\lambda) d\lambda \right) = \exp \left( - \iint s^* d\gamma \right) \quad (10)$$

where the last equality is obtained by Stokes' theorem and  $\iint$  denotes the integral over the two dimensional submanifold of  $M$  whose boundary is the loop  $m(t)$ ,  $t \in [0, 1]$ . Such a surface may not always exist in which case the last equality in (10) is dropped in the expression of the holonomy. For example, a circle which is one of the generators of a torus does not bound any surface on the torus. This case actually occurs in the dynamics of three coupled rigid bodies (see §5). Returning to the case when  $m(t)$  does bound a surface, the structure equations imply that  $d\gamma = D\gamma = \Omega$ , the curvature of  $\gamma$ . Thus, thinking of the curvature as a two-form on the base with values in the adjoint bundle, which for abelian groups is trivial, we obtain the well-known result:

**4.2 Corollary** *If  $\pi : E \rightarrow M$  is a principal  $G$ -bundle with  $G$  abelian and  $\gamma \in \Omega^1(E; \mathfrak{g})$  is a principal connection, then the holonomy of the closed path  $m(t)$  in  $M$  is given by the group element*

$$\text{holonomy} = \exp \left( - \iint d\gamma \right) = \exp \left( - \iint \Omega \right), \quad (11)$$

where the integral is taken over any two-dimensional submanifold in  $M$  whose boundary is  $m(t)$ .

For the  $S^1$ -connection on  $J^{-1}(\mu) \rightarrow T^*Q_\mu$  given in Proposition 2.2 the holonomy of a closed loop is given by

$$\exp \left( \frac{1}{\langle \mu, \xi \rangle} \iint \omega_\mu \right);$$

the integral being taken over the two-manifold in  $M$  whose boundary is the given loop. Returning to the reconstruction formula for the free rigid body given in Example 1F, note that the first term in (1F.2) is the holonomy of the closed integral curve on  $S^2$ .

## §5 Reconstruction Phases

In this section we give a number of examples of how to compute phases that arise from pure reconstruction (*i.e.*, without adiabaticity) using the theory of the preceding section. The free rigid body was already indicated in the introduction. Here we consider a number of other mechanical systems. In addition, we compute the rigid body phase using another choice of connection and compute the phase for the heavy top. In particular, we see that the nice choice of connection for the free rigid body is not so convenient for the heavy top, primarily because it does not drop down to configuration space. We also compute phases for some other examples, including coupled planar rigid bodies.

### §5A Quantum Mechanics

Classical holonomy can be used for the computation of the geometric Berry phase in quantum mechanics, using the well known fact that quantum mechanics is a special case of classical mechanics, and the Schrödinger equation is a special case of Hamilton's equations. These points are discussed in Abraham and Marsden [1978] and references therein. In this spirit, we have:

**5.1 Proposition** (Aharonov and Anandan [1987]) *The holonomy of a loop in projective complex Hilbert space is the exponential of twice the symplectic area of any two-dimensional submanifold whose boundary is the given loop.*

**Proof** Let  $\mathcal{H}$  be a complex Hilbert space and  $M = \mathbb{P}\mathcal{H}$ , the space of complex lines in  $\mathcal{H}$ . In Corollary 4.2, let  $E = \mathcal{H}$ ,  $G = S^1$  and use the connection  $A$  defined by

$$A(\psi) \cdot \varphi = \operatorname{Re} \langle -i\psi, \varphi \rangle = -\operatorname{Im} \langle \psi, \varphi \rangle$$

where  $\varphi, \psi \in \mathcal{H}$  and  $\langle \cdot, \cdot \rangle$  is the Hermitian inner product. The curvature is the differential of  $A$  and equals

$$\Omega(\varphi, \psi) = 2\operatorname{Im} \langle \varphi, \psi \rangle = -2\omega(\varphi, \psi)$$

where  $\omega \in \Omega^2(\mathcal{H})$  is the usual symplectic form on complex Hilbert space. Therefore  $-\iint \Omega = 2\iint \bar{\omega} =$  twice the symplectic area of the 2-manifold whose boundary is the given loop, where  $\bar{\omega}$  is the reduced symplectic form on projective Hilbert space  $\mathbb{P}\mathcal{H}$ . ■

### Remarks

**a** The result of Proposition 5.1 can be derived from Proposition 2.2 (generalized to exact symplectic manifolds). Take  $P = \mathcal{H}$  with the symplectic form  $\omega(\varphi, \psi) = -\operatorname{Im} \langle \varphi, \psi \rangle$ , and the  $S^1$ -action the multiplication of a vector by a scalar of the form  $e^{i\theta}$ . Then  $G = G_\mu = S^1$  for any  $\mu \in \mathfrak{g}^* = \mathbb{R}$ . The generator of  $\mathfrak{g}$  is taken to be 1. If  $\xi \in \mathbb{R}$ , then  $\xi_{\mathcal{H}}(\varphi) = i\xi\varphi$  for any  $\varphi \in \mathcal{H}$  so that the momentum map  $J : \mathcal{H} \rightarrow \mathbb{R}$  is  $J(\varphi) = -\|\varphi\|^2/2$ . The symplectic form  $\omega$  is exact and  $\omega = -d\theta$ , where  $\theta(\varphi) \cdot \psi = \frac{1}{2} \operatorname{Im} \langle \varphi, \psi \rangle$ . The level set  $J^{-1}\left(-\frac{1}{2}\right)$  is the sphere in  $\mathcal{H}$  of radius 1 and  $P_\mu = \mathbb{P}\mathcal{H}$ . Thus the connection  $A$  is precisely the one used in the proof of Proposition 5.1.

**b** The connection  $A$  used in the proof of 5.1 is the connection used in Simon [1985]. ♦

### §5B Phases for simple mechanical systems with $S^1$ -symmetry

Let  $(Q, \langle \cdot, \cdot \rangle)$  be a Riemannian manifold,  $G$  a Lie group acting by isometries on  $Q$  and assume  $H(\alpha_q) = \frac{1}{2} \|\alpha_q\|^2 + V(q)$  is a classical Hamiltonian, where  $V : Q \rightarrow \mathbb{R}$  is a  $G$ -invariant potential energy and  $\|\alpha_q\|$  denotes the norm of the bundle metric induced by  $\langle \cdot, \cdot \rangle$  on  $T^*Q$ , i.e.,  $\|\alpha_q\| = \|\alpha_q^\# \|$ , where  $\# : T^*Q \rightarrow TQ$  is the index raising operation induced by the metric. The momentum map  $J : T^*Q \rightarrow \mathfrak{g}^*$  is  $J(\alpha_q) \cdot \xi = \alpha_q \cdot \xi_q(q)$  for any  $\xi \in \mathfrak{g}$ . Let  $\mu \in \mathfrak{g}^*$  and assume throughout this section that  $G_\mu = S^1$  and that  $\rho_\mu : Q \rightarrow Q/G_\mu$  and  $\pi_\mu : J^{-1}(\mu) \rightarrow (T^*Q)_\mu$  are principal  $S^1$ -bundles. Let  $\zeta$  be a generator of  $\mathfrak{g}_\mu$  implementing the isomorphism of  $\mathfrak{g}_\mu$  with  $\mathbb{R}$ . Let  $\bar{c}_\mu(t)$  be an integral curve of the reduced system with Hamiltonian  $H_\mu$  on  $(T^*Q)_\mu$  and assume it is periodic with period  $T$ . If  $\bar{c}(t)$  denotes the reconstructed integral curve lying in the level set  $J^{-1}(\mu)$ , then

$$\bar{c}(t) = \exp(\varphi\zeta) \cdot \bar{c}(0) \quad (1)$$

for some angle  $\varphi$ , called the *total phase* of the integral curve  $\bar{c}(t)$ . In this section we shall compute  $\varphi$  in two ways, by choosing the connection  $A$  of Proposition 2.2 and the connection  $\gamma_{\text{mech}}^\mu$  of Corollary 2.4 where  $\gamma = \gamma_{\text{mech}}$  is the mechanical connection associated to the given metric. See the discussion surrounding Corollary 2.6.

We begin with the connection  $A \in \Omega^1(J^{-1}(\mu))$  given by

$$A = \frac{1}{\mu \cdot \zeta} \theta_\mu \quad (2)$$

where  $\theta_\mu$  is the pull-back of the canonical one-form  $\theta$  of  $T^*Q$  to  $J^{-1}(\mu)$ . Let  $d_A(t)$  denote the  $A$ -horizontal lift of  $\bar{c}_\mu(t)$  and  $q_A(t) = \pi(d_A(t))$  its base curve, where  $\pi : T^*Q \rightarrow Q$  is the canonical projection. By Theorem 2.1,  $c(t) = \exp(\psi(t)\zeta) \cdot d_A(t)$ , where  $\dot{\psi}(t) = (A \cdot X_H)(d_A(t)) = \theta(d_A(t)) \cdot X_H(d_A(t)) / (\mu \cdot \zeta)$ . Thus the base integral curve  $g(t)$  of  $c(t)$  equals  $g(t) := \pi(c(t)) = \exp(\psi(t)\zeta) \cdot d_A(t)$ , whence by  $G$ -invariance of  $V$ , we get  $V(q_A(t)) = V(q(t)) = \bar{V}_\mu(q_\mu(t))$ , where  $\bar{V}_\mu : Q/G_\mu \rightarrow \mathbb{R}$  is induced by  $V$  via  $V = \bar{V}_\mu \circ \rho_\mu$  and  $q_\mu(t)$  is the projection of  $\bar{c}_\mu(t) \in (T^*Q)_\mu$  to  $Q/G_\mu$  (see Proposition 2.2). Thus,

$$\theta(d_A(t)) \cdot X_H(d_A(t)) = \|d_A(t)\|^2 = 2H(d_A(t)) - 2V(q_A(t)) = 2H_\mu - 2\bar{V}_\mu(q_\mu(t))$$

where  $H_\mu$  is the constant value of the reduced Hamiltonian on the integral curve  $\bar{c}_\mu(t)$ . Therefore

$$\psi(t) = \frac{1}{\mu \cdot \zeta} \left( 2H_\mu t - 2 \int_0^t \bar{V}_\mu(q(s)) ds \right). \quad (3)$$

Since the curvature of the connection (2) is  $-\omega_\mu/(\mu \cdot \zeta)$ , where  $\omega_\mu$  is the reduced symplectic form on  $(T^*Q)_\mu$ , Corollary 4.2 shows that the holonomy is the exponential of

$$\left( \frac{1}{\mu \cdot \zeta} \iint_D \omega_\mu \right) \zeta \quad (4)$$

where  $D$  is a two-dimensional surface in  $(T^*Q)_\mu$  whose boundary is  $\bar{c}_\mu(t)$ . As we discussed in §4B such a surface might not always exist, but we shall assume it does, for convenience, in the sequel. Consequently, by the reconstruction method of Theorem 2.1, the total phase of the integral curve  $\bar{c}(t)$  in  $J^{-1}(\mu)$  equals

$$\phi = \frac{1}{\mu \cdot \zeta} \iint_D \omega_\mu + \frac{2(H_\mu - \langle \bar{V}_\mu \rangle)T}{\mu \cdot \zeta} \quad (5a)$$

where

$$\langle \bar{V}_\mu \rangle = \frac{1}{T} \int_0^T \bar{V}_\mu(q_\mu(t)) dt \quad (5b)$$

is the average of the potential  $V$  on the base integral curve  $q_\mu(t)$ . Note that if  $Q = \mathfrak{so}(3)$ ,  $V = 0$ , and  $H$  is the kinetic energy of a free rigid body, (5) reduces to formula (2) in §1G by choosing  $\zeta = \mu/\|\mu\|$  and where we have identified  $\mathfrak{so}(3)$  and its dual with  $\mathbb{R}^3$  in the usual manner (using the Killing form).

Next we turn to the total phase computation using the connection  $\gamma_{\text{mech}}^\mu$  induced by the mechanical connection  $\gamma_{\text{mech}}$ . We begin by translating the results of Corollary 3.6 to the



cotangent bundle setting. The notation will be that of §3C and §3D, namely,  $L$  denotes a classical Lagrangian with energy  $E(v_q) = \frac{1}{2} \|v_q\|^2 + V(q)$ ,  $\mathbb{FL} = \flat : TQ \rightarrow T^*Q$  is the corresponding Legendre transform which is a vector bundle isomorphism,  $c_\mu(t)$  is an integral curve of the reduced system on  $(TQ)_\mu$ , and  $c(t)$  is a reconstructed integral curve on the  $\mu$ -level set of the momentum map. We shall fix the mechanical connection  $\gamma_{\text{mech}} \in \Omega^1(Q)$  inducing both the connections  $\tilde{\gamma}_{\text{mech}}^\mu$  (in Corollary 2.4) and  $\gamma_{\text{mech}}^\mu$  (see (3) in §3C). In addition, we shall use in the cotangent bundle reduction theorem both in cotangent (Proposition 2.3) and tangent (Proposition 3.3) formulations, the embeddings into  $T^*(Q/G_\mu)$  and  $T(Q/G_\mu)$  respectively as being given by momentum or velocity shifts induced by  $\gamma$  (see closing comments in §3C). We let  $H = E \circ (\mathbb{FL})^{-1}$  be the corresponding Hamiltonian system on  $T^*Q$ . Since  $(\mathbb{FL})^* \tilde{\gamma}_{\text{mech}}^\mu = \gamma_{\text{mech}}^\mu$  as remarked in §3C,  $\mathbb{FL}$  will transport all information from the Lagrangian side to the Hamiltonian side. For example, if  $(\mathbb{FL})_\mu : (TQ)_\mu \rightarrow (T^*Q)_\mu$  is the induced diffeomorphism, we conclude that  $\bar{c}_\mu(t) = \mathbb{FL}(c_\mu(t))$  is an integral curve of the reduced Hamiltonian system on  $(T^*Q)_\mu$ , that  $\bar{c}(t) = \mathbb{FL}(c(t))$  is the reconstructed integral curve on  $J^{-1}(\mu)$ , and that  $\bar{d}(t) = \mathbb{FL}(d(t))$  is the  $\gamma_{\text{mech}}^\mu$ -horizontal lift of  $\bar{c}_\mu(t)$  iff  $d(t)$  is the  $\gamma_{\text{mech}}^\mu$ -horizontal lift of  $\bar{c}_\mu(t)$ . Thus by Corollary 3.6 we have

$$\bar{c}(t) = \exp(\theta(t)\zeta) \cdot \left( q'_h(t) + \frac{\mu \cdot \zeta}{\|\zeta_Q(q_h(t))\|^2} \zeta_Q(q_h(t)) \right)^\flat \quad (6)$$

with

$$\theta(t) = (\mu \cdot \zeta) \int_0^t \frac{ds}{\|\zeta_Q(q_h(s))\|^2}, \quad (7)$$

where  $q_h(t) \in Q$  is the  $\gamma_{\text{mech}}$ -horizontal lift of the projection  $q_\mu(t) \in Q/G_\mu$  of the reduced integral curve  $\bar{c}_\mu(t) \in (T^*Q)_\mu \subset T^*(Q/G_\mu)$ .

Let us determine the  $\tilde{\gamma}_{\text{mech}}^\mu$ -holonomy of the periodic curve  $\bar{c}_\mu(t)$  with period  $T$ . In the proof of Corollary 3.4 we showed that the  $\gamma_{\text{mech}}^\mu$ -horizontal lift of  $c_\mu(t)$  is the curve  $d(t) = q'_h(t) + (\mu \cdot \zeta) \zeta_Q(q_h(t)) / \|\zeta_Q(q_h(t))\|^2$  (see formula (4) in §3C). Thus since  $\bar{d}(t) = d(t)^\flat$  and the group action is by isometries, the holonomy of  $\bar{c}_\mu(t)$  measured from  $\bar{c}(0)$  equals the holonomy of  $c_\mu(t)$  measured from  $c(0)$ . Let  $k \in G_\mu = S^1$  be the holonomy of  $q_\mu(t)$  measured from  $q(0)$ , i.e.,

$$q_h(T) = k \cdot q_h(0) = k \cdot q(0). \quad (8)$$

Since  $G_\mu$  is abelian and  $\zeta \in \mathfrak{g}_\mu$ , the vector field  $\zeta_Q$  is equivariant and we get

$$\begin{aligned}
d(T) &= q_h'(T) + \frac{\mu \cdot \zeta}{\|\zeta_Q(q_h(t))\|^2} \zeta_Q(q_h(T)) \\
&= k \cdot q_h'(0) + \frac{\mu \cdot \zeta}{\|k \cdot \zeta_Q(q_h(0))\|^2} k \cdot \zeta_Q(q_h(0)) \\
&= k \cdot d(0)
\end{aligned} \tag{9}$$

i.e., the holonomy of  $\bar{c}_\mu(t)$  measured from  $c(0)$  equals the holonomy of  $q_\mu(t)$  measured from  $q(0) = \tau(c(0)) = \pi(\bar{c}(0))$ . Letting  $k = \exp(\chi\zeta)$ , we conclude from Theorem 2.1 and (7) that the total phase equals

$$\varphi = \chi + (\mu \cdot \zeta) \int_0^T \frac{ds}{\|\zeta_Q(q_h(s))\|^2}. \tag{10}$$

We collect formulas (5) and (10) in the following :

**5.2 Proposition** *Given is a simple mechanical system on  $T^*Q$  with Hamiltonian  $H$ . Assume that for  $\mu \in \mathfrak{g}^*$ ,  $G_\mu = S^1$ , that  $J^{-1}(\mu) \rightarrow (T^*Q)_\mu$  and  $Q \rightarrow Q/G_\mu$  are principal  $S^1$ -bundles and that  $\gamma_{\text{mech}} \in \Omega^1(Q)$  is the connection on the second bundle whose horizontal subbundle is the orthogonal complement of the vertical bundle. Let  $\bar{c}_\mu(t) \in J^{-1}(\mu)$  be a periodic integral curve with period  $T$  of the reduced system with energy  $H_\mu$  and let  $q_\mu(t) \in Q/G_\mu$  be its base integral curve. Then the total phase of the reconstructed integral curve on the level set  $J^{-1}(\mu)$  measured from  $\bar{c}(0)$  equals*

$$\varphi = \frac{1}{\mu \cdot \zeta} \iint_D \omega_\mu + \frac{2(H_\mu - \langle \bar{V}_\mu \rangle)T}{\mu \cdot \zeta} = \chi + (\mu \cdot \zeta) \int_0^T \frac{ds}{\|\zeta_Q(q_h(s))\|^2}$$

where  $\zeta$  is a generator of  $\mathfrak{g}_\mu$ ,  $\omega_\mu$  is the reduced symplectic form on  $(T^*Q)_\mu$ ,  $V$  is the potential energy of  $H$ ,  $\langle \bar{V}_\mu \rangle$  is the average over  $q_\mu(t)$  of the induced function  $\bar{V}_\mu : Q/G_\mu \rightarrow \mathbb{R}$ ,  $q_h(t)$  is the  $\gamma_{\text{mech}}$ -horizontal lift to  $Q$  of  $q_\mu(t)$ , and  $\chi$  is the  $\gamma_{\text{mech}}$ -holonomy of  $q_\mu(t)$  measured from  $q(0) := \pi(\bar{c}(0))$ . Here,  $D$  is a two-dimensional surface in  $(T^*Q)_\mu$  whose boundary is  $\bar{c}_\mu(t)$  and whose existence is assumed. The first terms in the two formulas are the geometric and the second the dynamic phases.

### §5C Phases for the Free Rigid Body

As we already remarked, if  $Q = \mathbf{SO}(3)$  is endowed with the left invariant Riemannian metric whose value at the identity is  $\langle x, y \rangle = x \cdot \mathbb{I} y$ ,  $\mathbb{I} = \text{diag}(I_1, I_2, I_3)$ ,  $I_i > 0$ ,  $i = 1, 2, 3$ ,  $V = 0$ , and  $\zeta = \mu / \|\mu\|$ , the total phase formula given by the first equality in Proposition 5.2 is equal to formula (2) in §1.F, namely

$$\varphi = -\Lambda + \frac{2H_\mu T}{\|\mu\|}, \quad (1)$$

where  $H_\mu$  is the energy of the solution curve  $\Pi(t)$  on the sphere,  $T$  is its period, and  $\Lambda$  is the oriented solid angle it bounds. We shall derive below a similar formula but using the connection  $\gamma_{\text{mech}}^\mu$  which is given by the second equality in Proposition 5.2.

We begin by explicitly computing the connection  $\gamma_{\text{mech}}$  on the principal  $S^1$ -bundle  $\rho_\mu : \mathbf{SO}(3) \rightarrow \mathbf{SO}(3)/S^1 \approx S^2$ ,  $\rho_\mu(g) = g^{-1}\mu$ . If  $u_g \in T_g \mathbf{SO}(3)$ , since  $\zeta$  is the generator of  $\mathfrak{g}_\mu$ ,  $\zeta = \mu / \|\mu\|$  we can write the horizontal plus vertical decomposition of  $u_g$  as

$$u_g = v_g + \lambda \hat{\zeta}_{\mathbf{SO}(3)}(g) = v_g + \lambda \hat{\zeta}_g.$$

Since  $v_g$  is horizontal, i.e., orthogonal in the Riemannian metric of  $\mathbf{SO}(3)$  to  $\hat{\zeta}_g$ , we conclude that  $\langle u_g, \hat{\zeta}_g \rangle = \lambda \langle \hat{\zeta}_g, \hat{\zeta}_g \rangle$ , or using left-invariance,  $\langle g^{-1}u_g, g^{-1}\hat{\zeta}_g \rangle = \lambda \langle g^{-1}\hat{\zeta}_g, g^{-1}\hat{\zeta}_g \rangle$ , whence  $\lambda = (g^{-1}u_g)^\sim \cdot \mathbb{I} g^{-1}\zeta / g^{-1}\zeta \cdot \mathbb{I} g^{-1}\zeta$ , where  $\sim : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$  is the inverse of the Lie algebra homomorphism

$$x \in \mathbb{R}^3 \mapsto \hat{x} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \in \mathfrak{so}(3).$$

Therefore, the connection form  $\gamma_{\text{mech}} \in \Omega^1(\mathbf{SO}(3))$  is given by

$$\gamma_{\text{mech}}(g)(u_g) = \frac{(g^{-1}u_g)^\sim \cdot \mathbb{I} g^{-1}\zeta}{g^{-1}\zeta \cdot \mathbb{I} g^{-1}\zeta}. \quad (2)$$

To compute the dynamic phase, we need to determine the horizontal lift  $g_h(t) \in \mathbf{SO}(3)$  of the closed path  $\Pi(t)$  on  $S^2$ , the sphere of radius  $\|\mu\|$ . By (1),  $g_h(t)$  is the horizontal lift of  $\Pi(t)$  iff

$$g_h(t)^{-1}\mu = \Pi(t) \text{ and } (g_h(t)^{-1}\dot{g}_h(t))^\sim \cdot \mathbb{I} g_h(t)^{-1}\zeta = 0. \quad (3)$$

However, the square of the norm of  $\hat{\zeta}_{\mathfrak{so}(3)}(g_h(t)) = \hat{\zeta}_{g_h(t)}$  in the  $\mathfrak{so}(3)$  metric is by left-invariance and (3),  $g_h(t)^{-1}\zeta \cdot \mathbb{I}g_h(t)^{-1}\zeta = \frac{1}{\|\mu\|^2} \Pi(t) \cdot \mathbb{I}\Pi(t)$ , so that the dynamic phase in the second formula of Proposition 5.2 equals

$$\|\mu\|^3 \int_0^T \frac{dt}{\Pi(t) \cdot \mathbb{I}\Pi(t)}. \quad (4)$$

To determine the geometric phase, we need to compute the curvature of the connection (2). On  $\mathfrak{so}(3)$  this equals  $d\gamma_{\text{mech}}$ . By Cartan's formula,  $d\gamma_{\text{mech}}(X, Y) = X[\gamma_{\text{mech}} \cdot Y] - Y[\gamma_{\text{mech}} \cdot X] - \gamma_{\text{mech}}[X, Y]$  for any two vector fields  $X, Y \in \mathfrak{X}(\mathfrak{so}(3))$ . If  $X(g) = T_e L_g \hat{x}$ ,  $x \in \mathbb{R}^3$  we have

$$(\gamma_{\text{mech}} \cdot X)(g) = \frac{(g^{-1} \cdot X(g)) \cdot \mathbb{I}g^{-1}\zeta}{g^{-1}\zeta \cdot \mathbb{I}g^{-1}\zeta} = \frac{x \cdot \mathbb{I}g^{-1}\zeta}{g^{-1}\zeta \cdot \mathbb{I}g^{-1}\zeta}$$

so that if  $Y(g) = T_e L_g \hat{y}$ , we get

$$\begin{aligned} Y[\gamma_{\text{mech}} \cdot X](g) &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} (\gamma_{\text{mech}} X)(g \exp \varepsilon \hat{y}) \\ &= - \frac{1}{(g^{-1}\zeta \cdot \mathbb{I}g^{-1}\zeta)^2} \left[ (x \cdot \mathbb{I} \hat{y} g^{-1}\zeta)(g^{-1}\zeta \cdot \mathbb{I}g^{-1}\zeta) - 2(x \cdot \mathbb{I}g^{-1}\zeta)(g^{-1}\zeta \cdot \mathbb{I} \hat{y} g^{-1}\zeta) \right]. \end{aligned}$$

Therefore

$$\begin{aligned} d\gamma_{\text{mech}}(g)(T_e L_g \hat{x}, T_e L_g \hat{y}) &= - \frac{1}{(g^{-1}\zeta \cdot \mathbb{I}g^{-1}\zeta)^2} \left\{ (\mathbb{I}y \cdot (x \times g^{-1}\zeta) - \mathbb{I}x \cdot (y \times g^{-1}\zeta) + (x \times y) \cdot \mathbb{I}g^{-1}\zeta) \right. \\ &\quad \left. - 2(y \cdot \mathbb{I}g^{-1}\zeta)((x \times g^{-1}\zeta) \cdot \mathbb{I}g^{-1}\zeta) + 2(x \cdot \mathbb{I}g^{-1}\zeta)((y \times g^{-1}\zeta) \cdot \mathbb{I}g^{-1}\zeta) \right\}. \quad (5) \end{aligned}$$

The curvature  $\Omega$  as a two-form on the base  $S^2$ , the sphere of radius  $\|\mu\|$ , is given by the condition  $\rho_\mu^* \Omega = d\gamma_{\text{mech}}$ . Since

$$T_g \rho_\mu(T_e L_g \hat{x}) = g^{-1}\mu \times x = \|\mu\| g^{-1}\zeta \times x, \quad (6)$$

we get  $\Omega(g^{-1}\mu)(g^{-1}\mu \times x, g^{-1}\mu \times y) = d\gamma_{\text{mech}}(g)(T_e L_g \hat{x}, T_e L_g \hat{y})$  so that letting  $\Pi = g^{-1}\mu$ , one has  $\|\Pi\| = \|\mu\|$  and (5) yields

$$\begin{aligned} \Omega(\Pi)(\Pi \times x, \Pi \times y) &= - \frac{\|\mu\|}{(\Pi \cdot \mathbb{I}\Pi)^2} \left\{ (\Pi \cdot \mathbb{I}\Pi)[\mathbb{I}y \cdot (x \times \Pi) - \mathbb{I}x \cdot (y \times \Pi) + (x \times y) \cdot \mathbb{I}\Pi] \right. \\ &\quad \left. - 2(y \cdot \mathbb{I}\Pi)((x \times \Pi) \cdot \mathbb{I}\Pi) + 2(x \cdot \mathbb{I}\Pi)((y \times \Pi) \cdot \mathbb{I}\Pi) \right\}. \end{aligned}$$

Up to a factor of 2, the last two terms equal

$$\begin{aligned}
 & ((\Pi \times \Pi) \cdot \mathbf{y})(\Pi \cdot \mathbf{x}) - ((\Pi \times \Pi) \cdot \mathbf{x})(\Pi \cdot \mathbf{y}) \\
 &= ((\Pi \times \Pi) \times \Pi) \cdot (\mathbf{y} \times \mathbf{x}) \\
 &= (\|\Pi\|^2 \Pi - (\Pi \cdot \Pi) \Pi) \cdot (\mathbf{x} \times \mathbf{y})
 \end{aligned}$$

so that  $\Omega$  becomes

$$\begin{aligned}
 \Omega(\Pi)(\Pi \times \mathbf{x}, \Pi \times \mathbf{y}) &= -\frac{\|\mu\|}{(\Pi \cdot \Pi)^2} \left\{ (\Pi \cdot \Pi) \Pi \cdot (\mathbf{y} \times \mathbf{x} - \mathbf{x} \times \mathbf{y}) \right. \\
 &\quad \left. - (\Pi \cdot \Pi)((\mathbf{x} \times \mathbf{y}) \cdot \Pi) + 2 \|\Pi\|^2 \Pi \cdot (\mathbf{x} \times \mathbf{y}) \right\}.
 \end{aligned}$$

However,

$$\Pi \cdot (\mathbf{y} \times \mathbf{x} - \mathbf{x} \times \mathbf{y}) - \Pi \cdot (\mathbf{x} \times \mathbf{y}) = -(\text{tr} \Pi) \Pi \cdot (\mathbf{x} \times \mathbf{y})$$

so that

$$\begin{aligned}
 \Omega(\Pi)(\Pi \times \mathbf{x}, \Pi \times \mathbf{y}) &= -\frac{\|\mu\|}{(\Pi \cdot \Pi)^2} \left[ 2\|\Pi\|^2 - (\Pi \cdot \Pi)(\text{tr} \Pi) \right] \Pi \cdot (\mathbf{x} \times \mathbf{y}) \\
 &= -\frac{2\|\Pi\|^2 - (\Pi \cdot \Pi)(\text{tr} \Pi)}{(\Pi \cdot \Pi)^2} dS(\Pi)(\Pi \times \mathbf{x}, \Pi \times \mathbf{y}) \quad (7)
 \end{aligned}$$

where  $dS(\Pi)(\Pi \times \mathbf{x}, \Pi \times \mathbf{y}) = \|\mu\| \Pi \cdot (\mathbf{x} \times \mathbf{y})$  is the area element on the sphere of radius  $\|\mu\|$ . Thus the total phase of the integral curve starting at  $c(0) \in J^{-1}(\mu)$ , reconstructed from the periodic orbit  $\Pi(t)$  of period  $T$  on the sphere of radius  $\|\mu\|$  is (mod  $2\pi$ ) equal to

$$\varphi = \iint_D \frac{2\|\Pi\|^2 - (\Pi \cdot \Pi)(\text{tr} \Pi)}{(\Pi \cdot \Pi)^2} ds + \|\mu\|^3 \int_0^T \frac{dt}{\Pi(t) \cdot \Pi(t)} \quad (8)$$

where  $D$  is the spherical cap bounded by  $\Pi(t)$ . The first term is the geometric and the second the dynamic phase. The right hand sides of (1) and (8) are equal since they both represent the total phase of the same integral curve. We see here how the split of the total phase into geometric and dynamic phases is entirely dependent on the chosen connection. The same phenomenon will be discussed for the ball in the hoop example (see §1B) in §12B.

### §5D Phases for the heavy top

The notations and conventions are as in the previous example, except that the Hamiltonian equals

$$H(\alpha_h) = \frac{1}{2} \|\alpha_h\|^2 + Mg\ell \mathbf{k} \cdot h\chi, \quad \alpha_h \in T_h^* \mathbf{SO}(3), \quad (1)$$

where  $\mathbf{k}$  is the unit vector of the spatial  $Oz$ -axis,  $g$  is the gravitational acceleration pointing in the negative direction of  $Oz$ ,  $M$  is the total mass of the body, the fixed point about which the body is moving is the origin, and  $\chi$  is the unit vector of the straight line segment of length  $\ell$  connecting the origin to the center of mass of the body. The Hamiltonian is left-invariant under the rotations about the spatial  $Oz$ -axis and the corresponding momentum map is  $\mathbf{J} : T^*\mathbf{SO}(3) \rightarrow \mathbb{R}$ ,  $\mathbf{J}(h, \Pi) = h\Pi \cdot \mathbf{k}$ , where  $T^*\mathbf{SO}(3)$  is identified with  $\mathbf{SO}(3) \times \mathbb{R}^3$  via left translations. The reduced spaces  $\mathbf{J}^{-1}(\mu)/S^1$  are generically cotangent bundles of spheres with the symplectic form equal to the canonical form plus a magnetic term, or, equivalently coadjoint orbits in  $\mathfrak{so}(3)^* = \mathbb{R}^3 \times \mathbb{R}^3$ ,  $O = \{(\Pi, \Gamma) \mid \Pi \cdot \Gamma = \mu, \|\Gamma\|^2 = 1\}$ ; the map  $\mathbf{J}^{-1}(\mu) \rightarrow O$  is given by  $\alpha_h \mapsto (T_e^*L_h(\alpha_h) = \Pi$ , where  $h^{-1}\mathbf{k} = \Gamma$ , as in Marsden, Ratiu, and Weinstein [1984]. After an appropriate momentum shift (see Proposition 2.3), the cotangent bundle projection  $\Pi : O \rightarrow S^2$  of  $O$  onto the sphere of radius 1, is given by  $\pi(\Pi, \Gamma) = \Gamma$ .

Let  $(\Pi(t), \Gamma(t))$  be a periodic orbit of period  $T$  of the heavy top equations. Then considering the connection  $\gamma \in \Omega^1(\mathbf{SO}(3))$  whose horizontal bundle is the orthogonal (relative to the rigid body metric) of the vertical bundle, all considerations of the previous example apply and we get the total phase formula

$$\varphi = \iint_D \frac{2\|\Pi\Gamma\|^2 - (\Gamma \cdot \Pi\Gamma)(\text{tr}\Pi)}{(\Gamma \cdot \Pi\Gamma)^2} ds + \int_0^T \frac{dt}{\Gamma \cdot \Pi\Gamma} \quad (2)$$

where  $D$  is the spherical cap on the unit sphere inclosed by the closed curve  $\Gamma(t)$ ,  $t \in [0, T]$ .

Next, let us recompute the total phase using the connection  $A$  of Proposition 2.2. Since in the left trivialization,  $\mathbf{J}^{-1}(\mu) = \{(h, \Pi) \mid h\Pi \cdot \mathbf{k} = \Pi \cdot \Gamma = \mu\}$  and  $\theta(h, \Pi)(h\hat{\mathbf{x}}, \lambda) = \Pi \cdot \mathbf{x}$ , the connection one-form (2) in this case becomes

$$A(h, \Pi)(h\hat{\mathbf{x}}, \lambda) = \frac{1}{\mu} \Pi \cdot \mathbf{x}. \quad (3)$$

The  $A$ -horizontal lift of the closed curve  $(\Pi(t), \Gamma(t))$  is  $(h(t), \Pi(t)) \in \mathbf{J}^{-1}(\mu)$ , where  $h(t)$  is uniquely determined by the conditions

$$h(t)^{-1}\mathbf{k} = \Gamma(t) \text{ and } (h(t)^{-1}\dot{h}(t))^\sim \cdot \Pi(t) = 0. \quad (4)$$

Therefore, the cotangent bundle projection of this horizontal lift is  $h(t)$  and hence

$$V(h(t)) = Mg\ell \mathbf{k} \cdot h\chi = Mg\ell \Gamma(t) \cdot \chi. \quad (5)$$

Thus, by Proposition 5.2, the dynamic phase is

$$\frac{1}{\mu} \left( 2H_{\mu} T - 2Mg\ell \int_0^t \Gamma(t) \cdot \chi dt \right). \quad (6)$$

The geometric phase is obtained in the following way. Let  $\mathcal{D}$  be a two-dimensional surface in  $T^*\mathbf{SO}(3)_{\mu} = O \cong TS^2$  bounded by the integral curve  $(\Pi(t), \Gamma(t))$  and let

$$\begin{aligned} \omega_{\mu}(\Pi, \Gamma)((\Pi \times \mathbf{x} + \Gamma \times \mathbf{y}, \Gamma \times \mathbf{x}), (\Pi \times \mathbf{x}' + \Gamma \times \mathbf{y}', \Gamma \times \mathbf{x}')) \\ = -\Pi \cdot (\mathbf{x} \times \mathbf{x}') - \Gamma \cdot (\mathbf{x} \times \mathbf{y}' - \mathbf{x}' \times \mathbf{y}) \end{aligned} \quad (7)$$

be the orbit symplectic form on  $O$ . Then the geometric phase equals

$$\frac{1}{\mu} \iint_{\mathcal{D}} \omega_{\mu}. \quad (8)$$

By (6) and (8), the total phase equals

$$\varphi = \frac{1}{\mu} \iint_{\mathcal{D}} \omega_{\mu} + \frac{1}{\mu} \left( 2H_{\mu} T - 2Mg\ell \int_0^T \Gamma(t) \cdot \chi dt \right). \quad (9)$$

Again, as in §5B, two distinct formulas, (2) and (9), are obtained for the total phase.

### §5E Phases for Coupled Planar Rigid Bodies

In this section we compute the phase for another mechanical system with  $S^1$  symmetry, namely a system of coupled rigid bodies in the plane. We thank P.S. Krishnaprasad for informing us of this example; see Krishnaprasad [1990] and Krishnaprasad and Yang [1990] for more information. Following Krishnaprasad's lecture at the Geometric Phases workshop at MSI-Cornell University (October 10-13, 1989), we first calculate the phase "by hand" without using any of the general theory, and then we shall show that the formula so obtained is a special case of formula (10) in §5B (see Proposition 5.2).

We consider  $n$  bodies forming a chain, as in Figure 5E-1. The center of mass motion has been assumed to have been eliminated, so the configuration space  $Q = T^n = S^1 \times \dots \times S^1$  is the product of  $n$  copies of the circle. The  $i^{\text{th}}$  circle describes the orientation of body  $i$  with

respect to an inertial frame. We let  $q^i \in S^1$  denote this angle for the  $i^{\text{th}}$  body and note that from these angles, the configuration of the bodies in the plane is known.

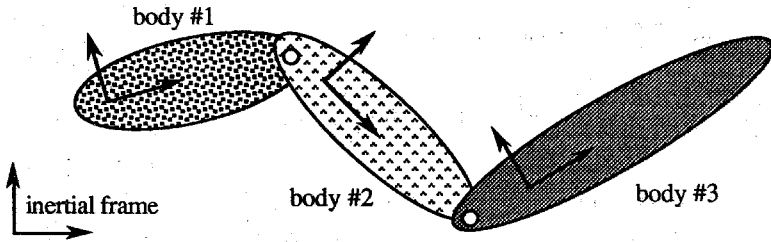


Figure 5E-1

As shown in Sreenath *et. al.*[1988], the equations for the dynamics of coupled planar rigid bodies are the Euler-Lagrange equations for a Lagrangian of the following form:

$$L = \frac{1}{2} \sum_{i,j=1}^n \omega^i J_{ij} \omega^j \quad (1)$$

where  $\omega^i = \dot{q}^i$ . The matrix  $J_{ij}$  is a positive definite inertia-type matrix that depends in a nontrivial way on the angles  $q^i$ . The Hamiltonian  $H : T^*Q \rightarrow \mathbb{R}$  corresponding to (1) is obtained by the usual Legendre transform,  $p_i = J_{ij} \omega^j$ , giving

$$H = \frac{1}{2} \sum_{i,j=1}^n p_i J^{ij} p_j \quad (2)$$

where  $J^{ij}$  is the inverse matrix of  $J_{ij}$ .

Imagine two types of motion. First, we consider free motion according to the Lagrangian or Hamiltonian system just described and second, the motion where the joints are controlled with a torque  $T_i$  exerted on body  $i$  by body  $i+1$  by, say, a motor at the joint between the bodies. Here,  $i = 1, \dots, n-1$  and the torques are *internal*, so in either case, the angular momentum of the overall system is conserved. Corresponding to overall rotations of the system, the group  $S^1$  acts on  $Q$  by the diagonal action  $\phi \cdot (q^1, q^2, \dots, q^n) = (q^1 + \phi, q^2 + \phi, \dots, q^n + \phi)$  and hence on the tangent and the cotangent spaces by the tangent and the cotangent lift. The Lagrangian and the Hamiltonian are both invariant under this action, as is reflected by the invariance of the matrix  $J$ . In fact,  $J$  depends only on the phase differences,  $\theta^i = q^{i+1} - q^i$ . Notice that the phase differences parametrize the *shape space*  $S = Q/S^1 = T^{n-1}$ . The reduced space is therefore  $T^*S$ . The unreduced motion occurs on the space of  $q^i$ 's and the reduced motion on the space of  $\theta^i$ 's. Note that for  $n = 3$ , the shape space is the two torus, so possible difficulties with a loop on  $S$  not



being the boundary of a surface in  $S$  can certainly occur. Identify the Lie algebra of  $S^1$  with  $\mathbb{R}$ ; if  $\xi \in \mathbb{R}$ , then the infinitesimal generator is given by  $\xi_Q(q) = \xi \sum_{i=1}^n \frac{\partial}{\partial q^i}$ . The associated momentum map is  $J(q, p)\xi = \xi \sum_{i=1}^n p_i = \xi e \cdot J\omega$ , where  $e$  is the column vector consisting of  $n$  1's. Write

$$\omega = \begin{bmatrix} \omega^1 \\ \omega^2 \\ \vdots \\ \omega^n \end{bmatrix} = \omega^1 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ \dot{\theta}_1 + \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_1 + \dots + \dot{\theta}_{n-1} \end{bmatrix} := \omega^1 e + M\dot{\theta}, \quad (2)$$

which defines the matrix  $M$ . Suppose that the unreduced motion takes place with angular momentum  $\mu = J(q, p)\xi = \xi e \cdot J\omega$ . Substitution from (2) gives

$$\mu = \xi e \cdot (J\omega_1 e) + \xi e \cdot JM\dot{\theta}$$

so that

$$\dot{q}^1 = \omega^1 = \frac{\mu}{\xi e \cdot (Je)} - \frac{JM\dot{\theta}}{\xi e \cdot (Je)} \quad (3)$$

Integration of (3) along a given curve  $c(t)$ ,  $0 \leq t \leq T$  in shape space gives

$$\Delta q^1 = \mu \int_0^T \frac{dt}{\xi e \cdot (Je)} - \int_0^T \frac{JM\dot{\theta}}{\xi e \cdot (Je)} \quad (4)$$

(A choice of  $\xi$  corresponds to a choice of the unit of time.) Note that the second term of (4) can be written as an integral over the shape space curve

$$\int_c \frac{JM d\theta}{\xi e \cdot (Je)} \quad (5)$$

where we regard  $JM d\theta$  as a one form on  $S$ . In particular, (5) is parametrization independent, which is a hallmark of a *geometric phase*. The first term of (4) corresponds to the *dynamic phase* and is parametrization dependent. Both integrals are regarded as integrals over a curve in shape space  $S$ . From (4) we can construct the change in  $q^1$ , and from this and the supposed known changes in the  $\theta^i$ , get the phase changes in the remaining variables, thus *reconstructing the dynamics explicitly in terms of quadratures*, with or without internal torques. We are, of course, most interested in these formulas when the curve in shape space is closed, but at this point it can be

a general curve. To be even more explicit, for the case of two bodies, and using the formula for  $J$  from Sreenath *et. al.* [1988], one finds that (5) equals

$$\int \frac{I_2 + \epsilon d_1 d_2 \sin \theta}{I_1 + I_2 + 2\epsilon d_1 d_2 \sin \theta} d\theta, \quad (6)$$

where  $\theta$  is the single joint angle,  $I_1$  and  $I_2$  are *augmented* moments of inertia of the two bodies (see Sreenath, *et. al.* [1988] for the definition),  $\epsilon = \frac{m_1 m_2}{m_1 + m_2}$  is the reduced mass and  $d_i$  are the distances from the center of mass to the hinge point for body  $i$ . We remark that (6) can be integrated explicitly (see for instance Frohlich [1979]), although the corresponding formula for  $n$  bodies is rather complicated.

Formula (4) can be derived from our Proposition 5.2. Indeed, we assert that *the formula (10) in Section 5B gives formula (4) above as a special case*. To see this, first note that the connection  $\gamma_{\text{mech}}$  is given from the remarks following Corollary 3.6 by

$$\gamma_{\text{mech}} = \frac{e \cdot J dq}{e \cdot J e} \quad (7)$$

Indeed, it is easily verified that (7) defines a connection one form and it is clear from its expression that its horizontal subspace (*i.e.*, the kernel of (7)) is the (metric)  $J$ -orthogonal complement to the  $S^1$ -orbits, that is to the space spanned by the vector  $e$ . The general formula from Proposition 5.2 is

$$\varphi = \chi + (\mu \cdot \zeta) \int_0^T \frac{ds}{\|\zeta_Q(q_h(s))\|^2}. \quad (8)$$

In this formula, the phase  $\varphi$  is the phase in the  $S^1$  fiber for the reconstructed motion. In our case, the projection  $Q \rightarrow S$  has a fiber which we identify with the phase of the first body, namely with  $\Delta q^1$ . By definition, the term  $\chi$  is the holonomy for the connection  $\gamma_{\text{mech}}$ . This is most easily computed using the formula (10) of §4.B. Indeed, here one interprets the matrix  $M$  as the derivative of a global section  $s$  for the bundle  $Q \rightarrow S$  and so the pull-back of the form (7) is just

$$s^* \gamma_{\text{mech}} = \frac{JM d\theta}{\xi e \cdot (J e)}$$

whose integral is the second term of (4). Thus, the second term of (4) is exactly the geometric phase, that is, the first term of (8). To deal with the second term of (8), take  $\zeta = \xi$  and use the fact that  $\xi_Q(q) = \xi \sum_{i=1}^n \frac{\partial}{\partial q^i} = \xi e$ , as we have seen. Thus,

$$\|\zeta_Q(q_h(s))\|^2 = \|\xi_Q(q_h(s))\|^2 = \xi^2 \|e\|^2 = \xi^2 e \cdot Je,$$

recalling that  $J$  depends on the joint angles only, so that this expression does not require the computation of the horizontal lift  $q_h$ , but depends only on the reduced variables. Thus, it is clear that the second term of (8) gives the first term of (4), so our assertion is proved.

When the reduced curve bounds a surface in  $S$  (which need not happen if the curve has non trivial homology class on the torus—that is, if it winds around the torus nontrivially) then the holonomy can be converted to a surface integral of the curvature of the mechanical connection.

Formula (5) of section 5B provides an alternative formula for the phase *in the case there are no torques*. (Torques are not allowed in that formula since it involves the energy of the reduced curve and with internal or external torques, this need not be conserved.) To include the possibility of torques, one replaces the term  $2H_\mu T$  with the integral of  $2H_\mu$  along the reduced curve. That is, we have

$$\Delta q^1 = \frac{1}{\mu\xi} \left( \iint_D \omega_\mu + 2\langle H_\mu \rangle T \right), \quad (9)$$

where

$$\langle H_\mu \rangle = \frac{1}{T} \int_0^T H_\mu(\theta(t)) dt.$$

In this formula, note that the reduced symplectic form is on  $T^*S = T^*T^{n-1}$  and the surface  $D$  is chosen to span the curve  $(\theta, p_\theta)$ , where  $p_\theta$  is the conjugate momentum to the joint velocities  $\dot{\theta}$ . Note that the computation of the double integral in (9) involves the magnetic terms of the reduction, which involves the computation of the curvature of the mechanical connection. Also, the reduced energy  $H_\mu$  is rather complicated. It is computed in Sreenath *et.al.* [1988]. One substitutes (3) in (1). This expression involves the amended potential for the reduction.

It seems, therefore, that for examples like coupled rigid bodies, the approach using the mechanical connection rather than the canonical one form connection  $A$  gives the most tractable results.

## §6 Averaging Connections

The purpose of this section is to define the Hannay-Berry connection by the averaging process. The main properties of such connections are given here and will be shown to characterize the connections in the next section.

### §6A Families of Actions

Let  $\pi : E \rightarrow M$  be a Poisson fiber bundle, i.e.,  $\pi$  is a surjective submersion, all fibers are Poisson manifolds, and the transition functions are Poisson maps. Let  $G$  be a Lie group. A **family of Hamiltonian  $G$ -actions** on  $E$  is a smooth (left)  $G$ -action on  $E$  such that each fiber  $\pi^{-1}(m)$  is invariant under the action and the action restricted to each fiber is Hamiltonian, i.e., it is Poisson and it admits a fiberwise momentum map  $I : E \rightarrow \mathfrak{g}^*$ . This means that for each  $\xi \in \mathfrak{g}$ , we have

$$\xi_E(p) = X_{I^\xi}(p) \quad (1)$$

where  $\xi_E(p) = \left. \frac{d}{dt} \right|_{t=0} (\exp t\xi \cdot p)$  is the infinitesimal generator of the action defined by  $\xi$  and  $X_{I^\xi}(p)$  is the Hamiltonian vector field on the fiber through  $p$  defined by the function  $I^\xi : E \rightarrow \mathbb{R}$  restricted to this fiber. Here,  $I^\xi$  denotes the real valued function defined by  $I^\xi(p) = I(p) \cdot \xi$ , for  $p \in E$  and  $\xi \in \mathfrak{g}$ . Since the action on each fiber is Hamiltonian, the symplectic leaves of the fiber are  $G$ -invariant. Also, note that the Casimirs are  $G$ -invariant.

An Ehresmann connection on  $\pi : E \rightarrow M$  is called **Poisson**, if its horizontal lift is a Poisson bracket derivation, i.e.,

$$(\text{hor } Z)[\{f, h\}] = \{(\text{hor } Z)[f], h\} + \{f, (\text{hor } Z)[h]\} \quad (2)$$

for all  $f, h : E \rightarrow \mathbb{R}$  and  $Z \in \mathfrak{X}(M)$ . Here, the bracket means that the functions are restricted to the fibers and the bracket is computed fiberwise. Equivalently, (2) says that

$$D\{f, h\} \cdot X = \{Df \cdot X, h\} + \{f, Dh \cdot X\} \quad (2')$$

for all  $X \in \mathfrak{X}(E)$ , where  $D$  is the covariant differentiation defined by  $\text{hor}$ . As we shall see later, our two main examples, the Cartan connection and the Hannay-Berry connection built from it (or from another suitable connection) do satisfy the condition (2).

If the fibers are symplectic, this notion is the same as that of a symplectic connection. Symplectic connections have been studied by Lichnerowicz [1978], Hess [1981], Marsden, Ratiu and Raugel [1990], and Guillemin and Lerman [1989].

At present, these two structures -- the connection and the  $G$ -action -- have no relationship to each other. The goal of this section is to show how a given connection can be altered so that

$$DI = 0 \quad (3)$$

holds. (In doing so, we may also have to alter  $I$  by adding to it fiberwise Casimirs.)

**6.1 Theorem: Averaging of Connections** *Let  $\pi : E \rightarrow M$  be a fiber bundle and  $\gamma \in \Omega^1(E, V)$  an Ehresmann connection. Suppose the compact Lie group  $G$  acts on  $E$  by bundle transformations, not necessarily covering the identity. Then the average  $\langle \gamma \rangle$  of  $\gamma$  is also an Ehresmann connection. Moreover the  $G$ -action commutes with the action of parallel translation with respect to  $\langle \gamma \rangle$ .*

**Proof** Since  $\gamma$  is vector-bundle valued we have to be careful in defining its average. If  $g$  is a bundle automorphism we define the *pull-back*  $g^*\gamma$  of  $\gamma$  by

$$(g^*\gamma)(e) \cdot v = Tg^{-1} \cdot \gamma(g \cdot e) \cdot Tg \cdot v.$$

This works because  $Tg$  maps  $V$  to  $V$ . Define the *average* of  $\gamma$  by

$$\langle \gamma \rangle = \frac{1}{|G|} \int_G (g^*\gamma) dg \quad (4)$$

where  $dg$  is a Haar measure and  $|G|$  is the total volume of  $G$ .

To check that the  $V$ -valued one form  $\langle \gamma \rangle$  is a connection one-form, we need only check that it is the identity on the vertical bundle  $V$ . Indeed, if  $v \in V$  then  $g^*\gamma v = v$ , so that

$$\langle \gamma \rangle \cdot v = \frac{1}{|G|} \int_G (g^*\gamma) \cdot v dg = \frac{1}{|G|} \int_G v dg = v.$$

One also checks that

$$g^*\langle \gamma \rangle = \langle \gamma \rangle \quad \text{for } g \in G.$$

It follows that  $G$  takes  $\langle \gamma \rangle$ -horizontal spaces to  $\langle \gamma \rangle$ -horizontal subspaces. Now let  $c(t)$  be a curve in  $M$  and  $\bar{c}$  a horizontal lift of  $c$ . It follows that  $g \cdot \bar{c}$  is a horizontal lift of  $g \cdot c$ . (Since  $G$  acts by bundle automorphisms it induces an action of  $G$  on  $M$ .) Thus the  $G$ -action commutes with parallel translation. (Note that this Proposition still holds if  $\pi : E \rightarrow M$  is just a submersion.) ■

### §6B The Hannay-Berry Connection

We now return to the setting where  $G$  defines a family of Hamiltonian  $G$ -actions. Fix a Poisson Ehresmann connection with horizontal lift  $\text{hor}_0$  and corresponding covariant differentiation operator  $D_0$ .

**6.2 Definition** *The Hannay-Berry (HB) connection induced by the Poisson-Ehresmann connection  $\text{hor}_0$  is the Ehresmann connection on  $\pi : E \rightarrow M$  obtained by averaging  $\text{hor}_0$  according to equation (4). We will let  $D$  denote its covariant derivative,  $\text{hor}$  its horizontal lift, and  $\gamma \in \Omega^1(E, \ker T\pi)$  its connection one-form.*

If  $\lambda$  is any tensor field defined along (as opposed to on) a  $G$ -invariant submanifold of  $E$ , its *average*  $\langle \lambda \rangle$  is the smooth  $G$ -invariant tensor field of the same type defined along the same submanifold by

$$\langle \lambda \rangle = \frac{1}{|G|} \int_G (\Phi_g^* \lambda) dg,$$

where  $\Phi : G \times E \rightarrow E$  is the  $G$ -action on  $E$ . Note that  $\langle \lambda \rangle$  is a  $G$ -invariant tensor field.

**6.3 Proposition** *Suppose  $G$  is compact and connected. Then the HB connection satisfies the following properties:*

- i *It is Poisson.*
- ii *If  $v \in T_m M$  then its HB horizontal lift is given by  $\text{hor } v = \langle \text{hor}_0 v \rangle$ .*
- iii  *$\text{hor } Z = \text{hor}_0 Z + X_{K \cdot Z}$ , for a smooth function  $K \cdot Z : E \rightarrow \mathbb{R}$  and  $Z \in \mathfrak{X}(M)$ .*
- iv *The connection one-form of the HB connection is given by*

$$\gamma(v) = \gamma_0(v) - X_{K \cdot T\pi(v)}(p), \text{ for } v \in T_p E. \quad (7)$$

- v  *$\langle K \cdot Z \rangle$  is a fiberwise Casimir function.*
- vi  *$D\langle \lambda \rangle = \langle D\lambda \rangle = \langle d\lambda \rangle \circ P_{\text{hor}}$ , for any  $\lambda \in \Omega^k(E)$ ,  $k = 0, 1, \dots$ , where  $P_{\text{hor}}$  is the horizontal projection relative to  $\text{hor}$ .*
- vii  *$D I^\xi \cdot \text{hor } Z = \langle D_0 I^\xi \cdot \text{hor } Z \rangle$  is a Casimir for each  $\xi \in \mathfrak{g}$  and  $Z \in \mathfrak{X}(M)$ . Moreover,  $DI = \langle D_0 I \rangle$  and hence  $DI = 0$  iff  $\langle D_0 I \rangle = 0$ .*

#### Remarks

1 The function  $K$  in property iii can, in many examples, be constructed using symmetry properties. See §8.

**2 Property vi** holds for any averaged connection. The rest of the properties are consequences of the following general principle:

If  $E$  has structure group  $G$ , and both  $\gamma_0$  and  $G$  preserve this structure, then  $\langle \gamma_0 \rangle$  also preserves this structure. In the HB case,  $G$  is the group of Hamiltonian automorphisms of the fiber.

Instead of trying to prove this general principle, we will prove proposition 6.3 "by hand." The proof of part iv of proposition 6.3 will follow from a lemma which we prove first. ♦

**6.4 Lemma** *Let  $\pi : E \rightarrow M$  be a Poisson fiber bundle and  $\text{hor}_0$  be the horizontal lift of a Poisson-Ehresmann connection. Then*

$$[\text{hor}_0 Z, X_f] = X_{df \cdot \text{hor}_0 Z}$$

*In particular, if  $E$  is also endowed with a family of Hamiltonian  $G$ -actions with parametrized momentum map  $I : E \rightarrow \mathfrak{g}^*$ , then*

$$[\text{hor}_0 Z, \xi_E] = X_{dI \xi \cdot \text{hor}_0 Z}$$

*for all  $Z \in \mathfrak{X}(M)$  and  $\xi \in \mathfrak{g}$ .*

**Proof** For any  $g : E \rightarrow \mathbb{R}$  we have

$$\begin{aligned} [\text{hor}_0 Z, X_f][g] &= (\text{hor}_0 Z)[X_f[g]] - X_f[(\text{hor}_0 Z)[g]] \\ &= (\text{hor}_0 Z)[\{g, f\}] - \{(\text{hor}_0 Z)[g], f\} \\ &= \{g, (\text{hor}_0 Z)[f]\} = X_{(\text{hor}_0 Z)[f]}[g] \end{aligned}$$

and the first formula is proved. The second follows from the first and the equality  $\xi_E = X_{I\xi}$ . ■

### Proof of 6.3

ii Let  $v \in T_m M$ .  $\text{hor } v$  is defined by the two conditions:  $T\pi \cdot \text{hor } v = v$ ,  $\langle \gamma \rangle \cdot \text{hor } v = 0$ . Since  $T\pi \cdot \text{hor } v = v$  and  $G$  is fiber-preserving,  $\langle \text{hor}_0 v \rangle$  automatically satisfies the first condition. Therefore

$$\langle \text{hor}_0 v \rangle = \text{hor}_0 v + Y \quad (5)$$

where  $Y$  is some vertical vector field on  $E_m$ . Since  $\langle \langle \text{hor}_0 v \rangle \rangle = \langle \text{hor}_0 v \rangle$ , we have  $\langle Y \rangle = 0$ .

We now check that  $\langle \text{hor}_0 v \rangle$  satisfies the second condition. It is automatically  $G$ -invariant:  $T\Phi_g \cdot \langle \text{hor}_0 v \rangle_e = \langle \text{hor}_0 v \rangle_{g \cdot e}$ . Thus

$$\begin{aligned}
\langle \gamma_0 \rangle_e \cdot \langle \text{hor}_0 v \rangle &= \frac{1}{|G|} \int_G T\Phi_g^{-1} \cdot \gamma(g \cdot e) \cdot T\Phi_g \langle \text{hor}_0 v \rangle_e dg \\
&= \frac{1}{|G|} \int_G T\Phi_g^{-1} \cdot \gamma_0(g \cdot e) \cdot \langle \text{hor}_0 v \rangle_{g \cdot e} dg \\
&= \frac{1}{|G|} \int_G T\Phi_g^{-1} \cdot \gamma_0(g \cdot e) \cdot ((\text{hor}_0 v)_{g \cdot e} + Y_{g \cdot e}) dg \\
&= \frac{1}{|G|} \int_G T\Phi_g^{-1} \cdot Y_{g \cdot e} dg = \langle Y \rangle = 0,
\end{aligned}$$

and the second condition is satisfied.

iii Since  $G$  is compact and connected, the exponential map is onto. We can write  $g = \exp \xi$  for some  $\xi \in \mathfrak{g}$ . By Lemma 6.4,

$$\begin{aligned}
\Phi_g^*(\text{hor}_0 Z) - \text{hor}_0 Z &= \int_0^1 \frac{d}{dt} \Phi_{\exp t\xi}^*(\text{hor}_0 Z) dt = \int_0^1 \Phi_{\exp t\xi}^* [\xi_E, \text{hor}_0 Z] dt \\
&= - \int_0^1 \Phi_{\exp t\xi}^* X_{dI\xi} \cdot \text{hor}_0 Z dt = - \int_0^1 X_{\Phi_{\exp t\xi}^* (dI\xi \cdot \text{hor}_0 Z)} dt = X_{f_Z(g)}
\end{aligned}$$

where

$$f_Z(g) = - \int_0^1 \Phi_{\exp t\xi}^* (dI\xi \cdot \text{hor}_0 Z) dt : E \rightarrow \mathbb{R},$$

a smooth function on  $E$ . The function  $f_Z$  depends on  $\xi \in \mathfrak{g}$ . Since there can be more than one way to write  $g = \exp \xi$ , let  $g = \exp \eta$  for some other  $\eta \in \mathfrak{g}$  and denote by  $h_Z(g)$  the corresponding smooth function on  $E$ . Therefore

$$\Phi_g^*(\text{hor}_0 Z) - \text{hor}_0 Z = X_{f_Z(g)} = X_{h_Z(g)}$$

i.e.,  $f_Z(g) - h_Z(g)$  is fiberwise a Casimir and in particular is  $G$ -invariant. Therefore

$$\begin{aligned}
Y = \text{hor } Z - \text{hor}_0 Z &= \langle \text{hor}_0 Z \rangle - \text{hor}_0 Z = \frac{1}{|G|} \int_G \Phi_g^*(\text{hor}_0 Z) dg - \text{hor}_0 Z \\
&= \frac{1}{|G|} \int_G (\Phi_g^*(\text{hor}_0 Z) - \text{hor}_0 Z) dg = \frac{1}{|G|} \int_G X_{f_Z(g)} dg = X_{K \cdot Z}
\end{aligned}$$



where  $K \cdot Z = \frac{1}{|G|} \int_G f_Z(g) dg$ . Since any two possible  $f_Z(g)$ 's differ by a Casimir, the difference of their averages is again a Casimir, so any two possible  $K \cdot Z$ 's differ by a Casimir, which makes  $X_{K \cdot Z}$  well defined.

**i** For each  $Z \in \mathfrak{X}(M)$ ,  $\text{hor}_0 Z$  and  $X_{K \cdot Z}$  are both Poisson derivations. Thus,  $\text{hor } Z$ , which is their sum by **iii**, is a Poisson derivation.

**iv** This is a restatement of **iii**.

**v** Since  $G$  acts canonically on the fiber and using the definition of the average,

$$0 = \langle \langle \text{hor}_0 Z \rangle - \text{hor}_0 Z \rangle = \langle \text{hor } Z - \text{hor}_0 Z \rangle = \langle X_{K \cdot Z} \rangle = X_{\langle K \cdot Z \rangle}.$$

Therefore  $\langle K \cdot Z \rangle$  is fiberwise a Casimir.

**vi** We prove the result for  $\lambda = f$ , a function on  $E$ , the proof for general  $k$ -forms being similar. Letting  $Z \in \mathfrak{X}(M)$  and using that  $\text{hor } Z$  is  $G$ -invariant, we get

$$\begin{aligned} D\langle f \rangle \cdot \text{hor } Z &= d \left( \frac{1}{|G|} \int_G \Phi_g^* f dg \right) \cdot \text{hor } Z = \frac{1}{|G|} \int_G \Phi_g^* (df \cdot \text{hor } Z) dg = \langle df \cdot \text{hor } Z \rangle \\ &= \frac{1}{|G|} \int_G \Phi_g^* (Df \cdot \text{hor } Z) dg = \left( \frac{1}{|G|} \int_G \Phi_g^* (Df) dg \right) \text{hor } Z = \langle Df \rangle \cdot \text{hor } Z. \end{aligned}$$

We have proved that  $D\langle f \rangle \cdot \text{hor } Z = \langle Df \rangle \cdot \text{hor } Z = \langle df \cdot \text{hor } Z \rangle$ . Since  $\Phi_g^* \xi_E = \xi_E$  for any  $\xi \in \mathfrak{g}$  and  $g \in G$ , we have

$$\langle Df \rangle \cdot \xi_E = \frac{1}{|G|} \int_G \Phi_g^* (Df) \cdot \xi_E dg = \frac{1}{|G|} \int_G \Phi_g^* (Df \cdot \xi_E) dg = 0$$

since  $Df \cdot \xi_E = 0$ . Therefore both  $D\langle f \rangle$  and  $\langle Df \rangle$  vanish for vertical vectors, proving the equality  $D\langle f \rangle = \langle Df \rangle$ . Finally, since  $\langle D_0 f \rangle \circ P_{\text{hor}}$  vanishes on vertical vector fields, the string of equalities in **vi** is proved, if we show that

$$\langle df \rangle \cdot \text{hor } Z = \langle Df \cdot \text{hor } Z \rangle = \langle df \cdot \text{hor } Z \rangle$$

for all  $Z \in \mathfrak{X}(M)$ . But this has been shown above.

**vii** Since  $\text{hor } Z$  is  $G$ -invariant,  $[\text{hor } Z, \xi_E] = 0$  for all  $\xi \in \mathfrak{g}$ . But  $\xi_E = X_{I\xi}$ , so by lemma 6.4 we have  $X_{Df \cdot \text{hor } Z} = 0$ . Thus  $Df \cdot \text{hor } Z$  is a fiberwise Casimir. Since  $G$  acts in a Hamiltonian fashion, it leaves fiberwise Casimirs invariant. Consequently by **iii**,

$$\begin{aligned}
DI^\xi \cdot \text{hor } Z &= \langle DI^\xi \cdot \text{hor } Z \rangle = \langle dI^\xi \cdot (\text{hor}_0 Z + X_{K \cdot Z}) \rangle \\
&= \langle D_0 I^\xi \cdot \text{hor}_0 Z \rangle + \langle dI^\xi \cdot X_{K \cdot Z} \rangle = \langle D_0 I \rangle \cdot \text{hor } Z - \langle \xi_E [K \cdot Z] \rangle \\
&= \langle D_0 I^\xi \rangle \cdot \text{hor } Z - \xi_E [\langle K \cdot Z \rangle] = \langle D_0 I^\xi \rangle \cdot \text{hor } Z.
\end{aligned}$$

Thus  $DI^\xi$  and  $\langle D_0 I^\xi \rangle$  coincide on horizontal vectors relative to the HB connection. Since both vanish on vertical vectors, we get  $DI^\xi = \langle D_0 I^\xi \rangle$ . ■

## §7 Existence, Uniqueness, and Curvature of the Hannay-Berry Connection

In this section we show that the HB connection is the unique Ehresmann connection on  $\pi : E \rightarrow M$  satisfying three conditions. We also calculate its curvature. As in §5, we assume  $E$  has a family of Hamiltonian  $G$ -actions with a parametrized momentum map  $I : E \rightarrow \mathfrak{g}^*$  and let  $\text{hor}_0$  denote the horizontal lift of a Poisson Ehresmann connection. The proofs in this section are almost verbatim from Montgomery [1988].

### §7A Characterization of the HB Connection

**7.1 Theorem** *Let  $\text{hor}$  be the horizontal lift of an Ehresmann connection on  $\pi : E \rightarrow M$  satisfying*

- a**  $DI = 0$ , where  $D$  is the covariant differentiation given by  $\text{hor}$ ; this says that parallel translation relative to  $\text{hor}$  preserves the level sets of  $I$ ;
- b**  $\text{hor } Z = \text{hor}_0 Z + X_{K \cdot Z}$  for a smooth function  $K \cdot Z : E \rightarrow \mathbb{R}$ , where  $Z \mapsto K \cdot Z$  is linear;
- c**  $\langle K \cdot v \rangle$  is a Casimir function on  $\pi^{-1}(m)$  where  $v \in T_m M$ ; replacing  $K \cdot v$  by  $K \cdot v - \langle K \cdot v \rangle$ , we can assume that this Casimir is zero.

Then

- i** such a connection is unique;
- ii** such a connection exists if and only if the 'adiabatic condition'

$$\langle D_0 I \rangle = 0 \tag{A}$$

holds, in which case it is the Hannay-Berry connection, and its covariant differentiation on functions is given by

$$Df \cdot u = D_0 f \cdot u + \{f, K \cdot T\pi(u)\} \tag{1}$$

for any  $u \in TE$ .

**Proof** **i** Suppose there are two horizontal lifts  $\text{hor}_1, \text{hor}_2$  satisfying **a**, **b** and **c**. By **b**,  $\text{hor}_1 v = \text{hor}_0 v + X_{K_1 \cdot v}$ ,  $\text{hor}_2 v = \text{hor}_0 v + X_{K_2 \cdot v}$ . Thus  $\text{hor}_1 v - \text{hor}_2 v = X_{K_1 \cdot v - K_2 \cdot v}$  so that by **a**,  $dI \cdot X_{K_1 \cdot v - K_2 \cdot v} = 0$ . Therefore, for each  $\xi \in \mathfrak{g}$ ,

$$0 = dI^\xi \cdot X_{K_1 \cdot v - K_2 \cdot v} = -\xi_E[K_1 \cdot v - K_2 \cdot v]$$

and thus  $K_1 \cdot v - K_2 \cdot v$  is  $G$ -invariant by connectedness of  $G$ . In particular  $K_1 \cdot v - K_2 \cdot v = \langle K_1 \cdot v - K_2 \cdot v \rangle = \langle K_1 \cdot v \rangle - \langle K_2 \cdot v \rangle = 0$  by **c**. Therefore  $K_1 \cdot v = K_2 \cdot v$  for any  $v \in TM$  and the two connections are equal.

ii For any  $\xi \in \mathfrak{g}$  and  $Z \in \mathcal{X}(M)$ , **b** gives

$$0 = DI^\xi \cdot \text{hor } Z = dI^\xi \cdot \text{hor}_0 Z + dI^\xi \cdot X_{K \cdot Z} = D_0 I^\xi \cdot \text{hor } Z - \xi_E[K \cdot Z]$$

so that using **c** and proceeding as in 6.3vi, we conclude

$$0 = \langle D_0 I^\xi \rangle \cdot \text{hor } Z - \xi_E[\langle K \cdot Z \rangle] = \langle D_0 I^\xi \rangle \cdot \text{hor } Z.$$

This shows that (A) is a necessary condition for existence.

Now assume, conversely, that  $\langle D_0 I \rangle = 0$ . The HB connection satisfies **a**, **b**, and **c** by proposition 6.3.

Formula (1) is an immediate consequence of 6.3iii.

**Remark** We call condition (A) the 'adiabatic condition' because in the context of a family of completely integrable systems this equality is the content of the classical adiabatic theorem. See also §9.

**7.2 Corollary** *If  $G$  is semisimple and  $I$  is equivariant the adiabatic condition (A) holds. Consequently, the HB connection satisfies property **a**, **b**, and **c** of Theorem 7.1.*

**Proof** By properties vi and vii of Proposition 6.3,  $DI = \langle DI \rangle = D\langle I \rangle$ . By equivariance  $\langle I \rangle^\xi = I^\xi$  for every  $\xi \in \mathfrak{g}$ . Now  $\langle \xi \rangle$  is an Ad-invariant vector. Since  $G$  is semi-simple this means  $\langle \xi \rangle = 0$ , consequently  $\langle I \rangle = 0$ , and thus  $DI = 0$ . (Recall that for semisimple Lie groups the adjoint representation is irreducible.) ■

**7.3 Corollary** *Suppose, for each  $v \in T_m M$  there is a function  $\tilde{K} \cdot v : \pi^{-1}(m) \rightarrow \mathbb{R}$  satisfying*

$$dI^\xi \cdot \text{hor}_0 v + \{I^\xi, \tilde{K} \cdot v\} = 0 \quad (2)$$

*for all  $\xi \in \mathfrak{g}$ . Set  $K \cdot v = \tilde{K} \cdot v - \langle \tilde{K} \cdot v \rangle$  and  $\text{hor } v = \text{hor}_0 v + X_{K \cdot v}$ . Assume  $\langle D_0 I \rangle = 0$ . Then  $\text{hor}$  defines the HB connection.*

**Proof** hor so defined clearly determines an Ehresmann connection satisfying properties **b** and **c** of 7.1. To prove property **a**, fix  $\xi \in \mathfrak{g}$  and compute for  $Z \in \mathfrak{X}(M)$

$$\begin{aligned} dI^\xi \cdot \text{hor } Z &= dI^\xi \cdot (\text{hor}_0 Z + X_{K \cdot Z}) = \{I^\xi, K \cdot Z\} + dI^\xi \cdot \text{hor}_0 Z \\ &= \{I^\xi, \tilde{K} \cdot Z\} + dI^\xi \cdot \text{hor}_0 Z - \{I^\xi, \langle \tilde{K} \cdot Z \rangle\} = \xi_E[\langle \tilde{K} \cdot Z \rangle] = 0 \end{aligned}$$

by G-invariance of  $\langle \tilde{K} \cdot Z \rangle$ . By uniqueness, hor defines the HB connection. ■

### §7B Curvature of the Hannay-Berry Connection

**7.4 Corollary** In addition to the hypothesis of 7.1, assume that the curvature  $\Omega_0$  of  $\text{hor}_0$  is Hamiltonian i.e.,  $\bar{\Omega}_0(Z_1, Z_2) = X_{K \cdot (Z_1, Z_2)}$  for some  $K \cdot (Z_1, Z_2) : E \rightarrow \mathbb{R}$ . Then the curvature  $\bar{\Omega} : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}_{\text{vert}}(E; M)$  of the HB connection is also Hamiltonian, and given by

$$\bar{\Omega}(Z_1, Z_2) = X_{-\langle \{K \cdot Z_1, K \cdot Z_2\} \rangle + \langle K \cdot (Z_1, Z_2) \rangle} \quad (3)$$

**Proof** By definition of curvature,

$$\bar{\Omega}(Z_1, Z_2) = \text{hor}[Z_1, Z_2] - [\text{hor } Z_1, \text{hor } Z_2] .$$

Write  $\text{hor } Z_i = \text{hor}_0 Z_i + X_{K \cdot Z_i}$ ,  $i = 1, 2$ , so that by Lemma 6.4,

$$\begin{aligned} [\text{hor } Z_1, \text{hor } Z_2] &= [\text{hor}_0 Z_1 + X_{K \cdot Z_1}, \text{hor}_0 Z_2 + X_{K \cdot Z_2}] \\ &= [\text{hor}_0 Z_1, \text{hor}_0 Z_2] - X_{d(K \cdot Z_1) \cdot \text{hor}_0 Z_2} + X_{d(K \cdot Z_2) \cdot \text{hor}_0 Z_1} - X_{\{K \cdot Z_1, K \cdot Z_2\}} \\ &= [\text{hor}_0 Z_1, \text{hor}_0 Z_2] + X_{F_{12}} \end{aligned} \quad (4)$$

where

$$F_{12} = -\{K \cdot Z_1, K \cdot Z_2\} + d(K \cdot Z_2) \cdot \text{hor}_0 Z_1 - d(K \cdot Z_1) \cdot \text{hor}_0 Z_2 . \quad (5)$$

Let us show that  $F_{12} - K \cdot (Z_1, Z_2)$  satisfies

$$dI^\xi \cdot \text{hor}_0[Z_1, Z_2] + \{I^\xi, F_{12} - K \cdot (Z_1, Z_2)\} = 0 \quad (6)$$

for all  $\xi \in \mathfrak{g}$ . Indeed

$$dI^\xi \cdot \text{hor}_0[Z_1, Z_2] + \{I^\xi, F_{12} - K \cdot (Z_1, Z_2)\} = dI^\xi \cdot (\text{hor}_0[Z_1, Z_2] - [\text{hor}_0 Z_1, \text{hor}_0 Z_2])$$

$$\begin{aligned}
& + dI^\xi \cdot [\text{hor}_0 Z_1, \text{hor}_0 Z_2] + \{I^\xi, F_{12}\} - \{I^\xi, K \cdot (Z_1, Z_2)\} \\
& = (\text{hor}_0 Z_1)[(\text{hor}_0 Z_2 - X_{K \cdot Z_2})[I^\xi]] - (\text{hor}_0 Z_2)[(\text{hor}_0 Z_1 - X_{K \cdot Z_1})[I^\xi]] + \{I^\xi, F_{12}\} \\
& = -(\text{hor}_0 Z_1)[\{I^\xi, K \cdot Z_2\}] + (\text{hor}_0 Z_2)[\{I^\xi, K \cdot Z_1\}] + \{I^\xi, F_{12}\} \\
& = -\{(\text{hor}_0 Z_1)[I^\xi] - X_{K \cdot Z_1}[I^\xi], K \cdot Z_2\} - \{I^\xi, (\text{hor}_0 Z_1)[K \cdot Z_2]\} + \\
& \quad \{(\text{hor}_0 Z_2)[I^\xi] - X_{K \cdot Z_2}[I^\xi], K \cdot Z_1\} + \{I^\xi, (\text{hor}_0 Z_2)[K \cdot Z_1]\} + \{I^\xi, F_{12}\} \\
& = \{\{I^\xi, K \cdot Z_1\}, K \cdot Z_2\} + \{\{K \cdot Z_2, I^\xi\}, K \cdot Z_1\} - \{I^\xi, (\text{hor}_0 Z_1)[K \cdot Z_2]\} \\
& \quad - (\text{hor}_0 Z_2)[K \cdot Z_1] - F_{12}\} \\
& = -\{\{K \cdot Z_1, K \cdot Z_2\}, I^\xi\} - \{I^\xi, \{K \cdot Z_1, K \cdot Z_2\}\} = 0
\end{aligned}$$

so (6) holds. Therefore, by 7.3,  $F_{12} - K \cdot (Z_1, Z_2) - \langle F_{12} - K \cdot (Z_1, Z_2) \rangle$  is the Hamiltonian function generating the HB connection for  $[Z_1, Z_2]$ , i.e.,

$$\text{hor}[Z_1, Z_2] = X_{F_{12} - \langle F_{12} \rangle} - X_{K \cdot (Z_1, Z_2) - \langle K \cdot (Z_1, Z_2) \rangle} + \text{hor}_0[Z_1, Z_2] .$$

By (4),

$$\begin{aligned}
\bar{\Omega}(Z_1, Z_2) &= \text{hor}[Z_1, Z_2] - [\text{hor } Z_1, \text{hor } Z_2] \\
&= X_{F_{12} - \langle F_{12} \rangle} - X_{K \cdot (Z_1, Z_2) - \langle K \cdot (Z_1, Z_2) \rangle} + \text{hor}_0[Z_1, Z_2] - [\text{hor}_0 Z_1, \text{hor}_0 Z_2] - X_{F_{12}} \\
&= X_{-\langle F_{12} \rangle} - X_{K \cdot (Z_1, Z_2)} + X_{\langle K \cdot (Z_1, Z_2) \rangle} + \bar{\Omega}_0(Z_1, Z_2) \\
&= X_{-\langle F_{12} \rangle + \langle K \cdot (Z_1, Z_2) \rangle} .
\end{aligned}$$

By 6.3vi and G-invariance of  $\text{hor } Z_1$ , the condition  $\langle K \cdot Z_2 \rangle = 0$  implies

$$\begin{aligned}
0 &= D\langle K \cdot Z_2 \rangle \cdot \text{hor } Z_1 = \langle D\langle K \cdot Z_2 \rangle \rangle \cdot \text{hor } Z_1 = d\langle \langle K \cdot Z_2 \rangle \cdot \text{hor } Z_1 \rangle \\
&= \langle d\langle K \cdot Z_2 \rangle \cdot \text{hor}_0 Z_1 + d\langle K \cdot Z_2 \rangle \cdot X_{K \cdot Z_1} \rangle ;
\end{aligned}$$

i.e.,

$$\langle d\langle K \cdot Z_2 \rangle \cdot \text{hor}_0 Z_1 \rangle = \langle \{K \cdot Z_1, K \cdot Z_2\} \rangle$$

and similarly

$$-\langle d\langle K \cdot Z_1 \rangle \cdot \text{hor}_0 Z_2 \rangle = \langle \{K \cdot Z_1, K \cdot Z_2\} \rangle .$$

Therefore

$$\begin{aligned}
\langle F_{12} \rangle &= -\langle \{K \cdot Z_1, K \cdot Z_2\} \rangle + \langle d\langle K \cdot Z_2 \rangle \cdot \text{hor}_0 Z_1 \rangle - \langle d\langle K \cdot Z_1 \rangle \cdot \text{hor}_0 Z_2 \rangle \\
&= -\langle \{K \cdot Z_1, K \cdot Z_2\} \rangle + \langle \{K \cdot Z_1, K \cdot Z_2\} \rangle + \langle \{K \cdot Z_1, K \cdot Z_2\} \rangle \\
&= \langle \{K \cdot Z_1, K \cdot Z_2\} \rangle . \quad \blacksquare
\end{aligned}$$

### §7C Hamiltonian One-forms

From Theorem 7.1, it follows that the HB connection is uniquely determined by the one-form on  $M$  given by  $Z \in \mathfrak{X}(M) \mapsto K \cdot Z \in$  functions on  $E$  modulo fiber-Casimirs. This function is in general not  $G$ -invariant since  $\text{hor}_0$  isn't. We shall call this one-form the *Hamiltonian one-form of the HB connection*.

**7.5 Theorem** *Let  $\pi : E \rightarrow M$  be a Poisson fiber bundle having a family of Hamiltonian  $G$ -actions with parametrized momentum map  $I : E \rightarrow \mathfrak{g}^*$  and Poisson-Ehresmann connection  $\text{hor}_0$ .*

*Assume that the curvature  $\Omega_0$  of  $\text{hor}_0$  is Hamiltonian, i.e.,  $\bar{\Omega}(Z_1, Z_2) = X_{K \cdot \langle Z_1, Z_2 \rangle}$  for all  $Z_1, Z_2 \in \mathfrak{X}(M)$ . Let  $L(\mathfrak{g}, C(E))$  be the vector space of linear maps of the Lie algebra  $\mathfrak{g}$  of  $G$  to the space  $C(E)$  of fiberwise Casimir functions. Then  $\langle D_0 I \rangle$  is the pull-back of a closed  $L(\mathfrak{g}, C(E))$ -valued one-form on the base  $M$ .*

**Proof** We show that  $\langle D_0 I^\xi \rangle$  is a closed form for any  $\xi \in \mathfrak{g}$ . First, we prove that  $\langle D_0 I^\xi \rangle$  vanishes on vertical vectors  $u \in \ker T_p \pi$ . Indeed,

$$\langle D_0 I^\xi \rangle(p) \cdot u = \frac{1}{|G|} \int_G \Phi_g^* \langle D_0 I^\xi \rangle(p) \cdot u \, dg = \frac{1}{|G|} \int_G dI^\xi(\Phi_g(p)) \cdot P_{\text{hor}_0}(T_p \Phi_g(u)) \, dg, \quad (7)$$

where  $P_{\text{hor}_0}$  is the horizontal projection defined by  $\text{hor}_0$ . Since  $u$  is vertical, so is  $T_p \Phi_g(u)$  because  $\pi \circ \Phi_g = \pi$ . Therefore  $P_{\text{hor}_0}(T_p \Phi_g(u)) = 0$ . Thus the integrand in (7) is zero and hence  $\langle D_0 I^\xi \rangle(p) \cdot u = 0$ .

Second, we show that  $d\langle D_0 I \rangle$  vanishes on horizontal lifts. By 6.3vii, for  $Z_1, Z_2 \in \mathfrak{X}(M)$  we have

$$\begin{aligned} d\langle D_0 I^\xi \rangle(\text{hor } Z_1, \text{hor } Z_2) &= (\text{hor } Z_1)[\langle D_0 I^\xi \rangle \cdot \text{hor } Z_2] - (\text{hor } Z_2)[\langle D_0 I^\xi \rangle \cdot \text{hor } Z_1] \\ &\quad - \langle D_0 I^\xi \rangle[\text{hor } Z_1, \text{hor } Z_2] \\ &= (\text{hor } Z_1)[dI^\xi \cdot \text{hor } Z_1] - (\text{hor } Z_2)[dI^\xi \cdot \text{hor } Z_1] \\ &= (\text{hor } Z_1)[dI^\xi \cdot \text{hor } Z_2] - (\text{hor } Z_2)[dI^\xi \cdot \text{hor } Z_1] \\ &= d(dI^\xi)(\text{hor } Z_1, \text{hor } Z_2) - dI^\xi \cdot \Omega(\text{hor } Z_1, \text{hor } Z_2) \\ &= \{I^\xi, \langle \{K \cdot Z_1, K \cdot Z_2\} \rangle - \langle K \cdot Z_1, Z_2 \rangle\} = 0 \end{aligned}$$

by  $G$ -invariance of the  $\langle \{K \cdot Z_1, K \cdot Z_2\} \rangle - \langle K \cdot Z_1, Z_2 \rangle$  and conservation of momentum maps.

By step one,  $d\langle D_0 I^\xi \rangle$  vanishes on any pair of vertical vector fields. Indeed, if  $V_1, V_2$  are vertical vector fields, then so is  $[V_1, V_2]$  and thus

$$d\langle D_0 I^\xi \rangle(V_1, V_2) = V_1[\langle D_0 I^\xi \rangle \cdot V_2] - V_2[\langle D_0 I^\xi \rangle \cdot V_1] - \langle D_0 I^\xi \rangle \cdot [V_1, V_2] = 0.$$

Finally, let us show that  $d\langle D_0 I^\xi \rangle$  vanishes on a pair formed by a horizontal and a vertical vector field. Indeed, for  $Z \in \mathfrak{X}(M)$  and  $V \in \mathfrak{X}_{\text{vert}}(E; M)$ ,

$$\begin{aligned} d\langle D_0 I^\xi \rangle(\text{hor } Z, V) &= (\text{hor } Z)[\langle D_0 I^\xi \rangle \cdot V] - V[\langle D_0 I^\xi \rangle \cdot \text{hor } Z] - \langle D_0 I^\xi \rangle \cdot [\text{hor } Z, V] \\ &= -V[dI^\xi \cdot \text{hor } Z] = -V[dI^\xi \cdot \text{hor } Z]. \end{aligned}$$

By 6.4,  $X_{dI^\xi \cdot \text{hor } Z} = [\text{hor } Z, \xi_E] = 0$  since  $\text{hor } Z$  is  $G$ -invariant. Therefore  $dI^\xi \cdot \text{hor } Z$  is a Casimir function on every fiber, i.e., as a function of  $E$  with values in  $C(E)$ , it is constant. Consequently, as a  $C(E)$ -valued one-form,  $d\langle D_0 I^\xi \rangle(\text{hor } Z, V) = 0$ .

These four steps prove that:

- i  $d\langle D_0 I^\xi \rangle = 0$
- ii  $\langle D_0 I^\xi \rangle$  is a one-form on  $E$  vanishing on vectors that are tangent to the fibers and  $\langle D_0 I^\xi \rangle \cdot \text{hor } Z = dI^\xi \cdot \text{hor } Z$  is constant as a  $C(E)$ -valued function on  $E$ , i.e.,  $\langle D_0 I^\xi \rangle$  is the pull-back of a one-form  $\alpha_\xi$  on  $M$  with values in  $C(E)$ .

Since  $\pi$  is a surjective submersion, ii implies that  $d\alpha_\xi = 0$ . ■

**7.6 Corollary** *In a simply connected neighborhood  $U$  of any point  $m_0 \in M$  there is a  $L(\mathfrak{g}, C(E))$ -valued function  $f$  on  $U$  such that  $I' = (I + f \circ \pi) \mid \pi^{-1}(U)$  satisfies  $\langle D_0 I' \rangle = 0$ .*

**Proof** Let  $\alpha$  denote the  $L(\mathfrak{g}, C(E))$ -valued one-form on  $M$  whose pull-back is  $\langle D_0 I \rangle$  and define

$$f(m) = - \int_{m_0}^m \alpha$$

where the integral is taken over any path connecting  $m_0$  to an arbitrary point  $m \in U$ . By simple connectivity of  $U$ ,  $f$  is well defined, and so  $df = -\alpha$  on  $U$ . Then  $D_0(f \circ \pi) = d\pi^*f = \pi^*df = -\alpha = -\langle D_0 I \rangle$ . ■

In particular, if  $M$  is simply connected, the parametrized momentum map  $I$  can always be chosen such that  $\langle D_0 I \rangle = 0$ . Thus, for Poisson fiber bundles with simply connected base, the HB connection preserves the level sets of a carefully chosen parametrized momentum map of the Hamiltonian  $G$ -action.



## §8 The Hannay-Berry Connection in the Presence of Additional Symmetry

In this section we use symmetry to simplify the computation of the HB connection. Letting  $\pi : E \rightarrow M$ ,  $I : E \rightarrow \mathfrak{g}^*$ , and  $\text{hor}_0$  be as in the previous sections, assume there is another Lie group  $H$  acting on the *left* on  $E$  leaving  $I$  invariant, i.e.,  $I(h \cdot p) = I(p)$  for all  $h \in H$  and  $p \in E$ . If  $\eta \in \mathfrak{h}$ , the Lie algebra of  $H$ , the vertical projection  $\eta_E - P_{\text{hor}_0} \eta_E$  is a vector field tangent to the fibers of  $E$ ;  $P_{\text{hor}_0}$  denotes, as usual, the horizontal projection relative to  $\text{hor}_0$ . We say that the  $H$ -action is **Hamiltonian** if there is a function  $J : E \rightarrow \mathfrak{h}^*$ , called the **momentum mapping** such that

$$\eta_E - P_{\text{hor}_0} \eta_E = X_{J\eta} \quad (1)$$

where  $J^\eta(p) = J(\eta)(p)$ . In particular, the  $G$ -action is Hamiltonian.

In the examples,  $H$  acts by bundle transformations on  $E$ . This means that the action covers an action on  $M$ . In the examples, the group  $G$  is isomorphic to the isotropy subgroup of a point  $m$  of  $M$  i.e.,  $G = H_m$ .

In  $I^\xi(h \cdot p) = I^\xi(p)$ , let  $\xi \in \mathfrak{g}$ ,  $h = \exp t\eta$ , and take the time derivative at  $t=0$  to get

$$\begin{aligned} 0 &= dI^\xi(p) \cdot \eta_E(p) = dI^\xi(p) \cdot X_{J\eta}(p) + dI^\xi(p) \cdot P_{\text{hor}_0} \eta_E(p) \\ &= dI^\xi(p) \cdot P_{\text{hor}_0} \eta_E(p) + \{I^\xi, J^\eta\}(p) . \end{aligned}$$

By 7.3,

$$K \cdot v_\eta = J^\eta - \langle J^\eta \rangle , \quad (2)$$

where  $v_\eta = T_p \pi(\eta_E(p)) \in T_{\pi(p)} M$  has horizontal lift relative to  $\text{hor}_0$  equal to  $P_{\text{hor}_0} \eta_E(p)$ . This will determine the Hannay-Berry connection if  $T_{\pi(p)} M = T_p \pi(\{\eta_E(p) \mid \eta \in \mathfrak{g}\})$ . We have proved:

**8.1 Proposition** *Let  $\pi : E \rightarrow M$  be a Poisson fiber bundle with a family of Hamiltonian  $G$ -actions with parametrized momentum map  $I : E \rightarrow \mathfrak{g}^*$  and with a Poisson-Ehresmann connection  $\text{hor}_0$ . Assume  $\langle D_0 I \rangle = 0$ . Let another Lie group  $H$  act on  $E$  in a Hamiltonian fashion with momentum map  $J : E \rightarrow \mathfrak{h}^*$ . Assume that  $H$  leaves  $I$  invariant and that the tangent spaces to the  $H$ -orbits on  $E$  project by  $T\pi$  onto the corresponding tangent space of  $M$ . Then the HB connection satisfies (2).*

As in Montgomery [1988], formula (2) is useful for computing the HB connection for many interesting systems. We now set the stage to illustrate the usefulness of this idea; the results

will be summarized in Proposition 8.2. Assume that  $E = P \times O_\zeta$ , where  $O_\zeta$  is an adjoint orbit of a compact Lie group  $G$  and  $\zeta$  is a regular element, so the adjoint isotropy subgroup  $G_\zeta$  is a maximal torus in  $G$ . Assume  $P$  is a Poisson manifold and endow  $E$  with the trivial connection whose horizontal subbundle is  $0 \times O_\zeta \subset T(P \times O_\zeta)$ . This connection is a Poisson-Ehresmann connection on  $P \times O_\zeta$ . Let  $\langle, \rangle$  be a bi-invariant metric on  $\mathfrak{g}$  and use it to identify  $\mathfrak{g}^*$  and  $\mathfrak{g}$ . Assume that the  $G$ -action has an *equivariant* momentum map  $J : P \rightarrow \mathfrak{g}^*$  and define  $I : P \times O_\zeta \rightarrow \mathfrak{g}_\zeta$  by

$$\langle I(p, \eta), \lambda \rangle = \langle J(p), i_\eta(\lambda) \rangle, \quad (3)$$

where  $\lambda \in \mathfrak{g}_\zeta$  and  $i_\eta : \mathfrak{g}_\zeta \rightarrow \mathfrak{g}$  is given by  $i_\eta(\lambda) = \text{Ad}_h \lambda$ , where  $h \in G_\zeta$  is chosen to satisfy  $\eta = \text{Ad}_h \zeta$ . Such an  $h \in G_\zeta$  always exists since  $\eta \in O_\zeta$  and the definition of  $i_\eta$  is independent of  $h$  since  $G_\zeta$  is abelian. Since  $G_\zeta$  is a torus, the functions  $p \mapsto \langle I(p, \eta), \xi \rangle$ ,  $\xi \in \mathfrak{g}_\zeta$  generate a torus action on  $P$ , depending smoothly on  $\eta$ . This action, together with the trivial action on  $O_\zeta$  defines an action of  $G_\zeta$  on  $P \times O_\zeta$  with parametrized momentum map  $I$ . Since  $i_{g \cdot \eta}(\lambda) = g \cdot i_\eta(\lambda)$  for any  $\lambda \in \mathfrak{g}_\zeta$  and  $g \in G$ , the bi-invariance of the metric  $\langle, \rangle$  and the *equivariance* of  $J$  imply that  $I$  is *invariant under the  $G$ -action on  $P \times O_\zeta$* ; the momentum map of this action is  $(p, \eta) \mapsto J(p) - \eta$ .

We can apply Proposition 8.1 to this situation, with  $H$  there replaced by  $G$ , and  $G$  by  $G_\zeta$ . We begin by verifying the adiabatic condition  $\langle D_0 I \rangle = 0$ . Since any vector tangent to  $O_\zeta$  at  $h$  is of the form  $[\xi, \eta]$  for some  $\xi \in \mathfrak{g}$ , then for  $\lambda \in \mathfrak{g}_\zeta$  and  $v_p \in T_p P$ , we get

$$\langle D_0 I(p, \eta) \cdot (v_p, [\xi, \eta]), \lambda \rangle = \langle dI(p, \eta) \cdot (0, [\xi, \eta]), \lambda \rangle = \langle J(p), [\xi, i_\eta(\lambda)] \rangle = \langle \text{Ad}_{h^{-1}}[J(p), \xi], \lambda \rangle,$$

whence

$$D_0 I(p, \eta) \cdot (v_p, [\xi, \eta]) = \text{Ad}_{h^{-1}} P_\zeta[J(p), \xi] \quad (4)$$

where  $\text{Ad}_h \zeta = \eta$  and  $P_\zeta : \mathfrak{g} \rightarrow \mathfrak{g}_\zeta$  is the orthogonal projection relative to the metric  $\langle, \rangle$ . Taking the  $G_\zeta$ -average of (4), we get

$$\langle D_0 I \rangle(p, \eta) \cdot (v_p, [\xi, \eta]) = \langle P_\zeta[J(p), \xi] \rangle. \quad (5)$$

Writing  $\xi = \xi_1 + \xi_2$ , where  $\xi_1 \in \mathfrak{g}_\zeta$  and  $\xi_2 \in \mathfrak{g}_\zeta^\perp$ , we conclude that  $[J(p), \xi] = [J(p), \xi_2]$ , since  $\mathfrak{g}_\zeta$  is abelian and both  $J(p)$  and  $\xi_1$  belong to  $\mathfrak{g}_\zeta$ . Therefore,  $P_\zeta[J(p), \xi_2] = 0$  since  $[\mathfrak{g}_\zeta, \mathfrak{g}_\zeta^\perp] \subset \mathfrak{g}_\zeta^\perp$  and we conclude that  $\langle D_0 I \rangle = 0$ .

By (2), the Hamiltonian one-form of the HB connection for  $p \in P$ ,  $\eta \in O_\zeta$  and  $\lambda \in \mathfrak{g}$ , is given by

$$(K \cdot \lambda_{O_\zeta})(p, \eta) = J^\lambda(p, \eta) - \langle J^\lambda \rangle(p, \eta). \quad (6)$$

However

$$\begin{aligned}
 \langle J^\lambda \rangle(p, \eta) &= \frac{1}{|G_\eta|} \int_{G_\eta} (\Phi_g^* J^\lambda)(p, \eta) dg = \frac{1}{|G_\eta|} \int_{G_\eta} J^\lambda(g \cdot p, \text{Ad}_g \eta) dg \\
 &= \frac{1}{|G_\eta|} \int_{G_\eta} J^\lambda(g \cdot p) dg - \eta = \left\langle J(p), \frac{1}{|G_\eta|} \int_{G_\eta} \text{Ad}_{g^{-1}} \lambda dg \right\rangle - \eta.
 \end{aligned}$$

If  $\lambda = \lambda_1 + \lambda_2$  where  $\lambda_1 \in \mathfrak{g}_\eta$  and  $\lambda_2 \in \mathfrak{g}_\eta^\perp$ , then  $\text{Ad}_{g^{-1}} \lambda_1 = \lambda_1$  and  $\text{Ad}_{g^{-1}} \lambda_2 \in \mathfrak{g}_\eta^\perp = \mathfrak{g}_\zeta^\perp$ , so that

$$\langle J^\lambda \rangle(p, \eta) = \langle J(p), \lambda_1 \rangle - \eta. \quad (7)$$

Therefore, by (6) and (7),

$$(K \cdot \lambda_{O_\zeta}(\eta))(p, \eta) = \langle J(p), \mathbb{P}_\eta^\perp \lambda \rangle \quad (8)$$

where  $\mathbb{P}_\eta^\perp: \mathfrak{g} \rightarrow \mathfrak{g}_\eta^\perp$  is the orthogonal projection relative to the metric  $\langle \cdot, \cdot \rangle$ . Therefore by 6.3 the horizontal lift of the HB connection has the expression

$$(\text{hor } \lambda_{O_\zeta}(\eta))(p, \eta) = \left( X_{J \cdot \mathbb{P}_\eta^\perp(\lambda)}, \lambda_{O_\zeta}(\eta) \right) \quad (9)$$

and the connection one-form is

$$\gamma(p, \eta) \cdot (v_p, \lambda_{O_\zeta}(\eta)) = v_p - X_{J \cdot \mathbb{P}_\eta^\perp(\lambda)}(p), \quad (10)$$

for all  $v_p \in T_p P$  and  $\lambda_{O_\zeta}(\eta) \in T_\eta O_\zeta$ . The curvature is given by the Hamiltonian vector field whose Hamiltonian is the average of  $\{K \cdot \lambda_{O_\zeta}(\eta), K \cdot \lambda'_{O_\zeta}(\eta)\} = \{J \cdot \mathbb{P}_\eta^\perp \lambda, J \cdot \mathbb{P}_\eta^\perp \lambda'\} = J[\mathbb{P}_\eta^\perp \lambda, \mathbb{P}_\eta^\perp \lambda']$  (see Corollary 7.4), i.e.,

$$\bar{\Omega}(\lambda_{O_\zeta}(\eta), \lambda'_{O_\zeta}(\eta))(p, \eta) = - \left( X_{J[\mathbb{P}_\eta^\perp \lambda, \mathbb{P}_\eta^\perp \lambda']}(p), 0 \right). \quad (11)$$

Finally, let us compute the holonomy of a given closed path  $\eta(t)$  in  $O_\zeta$ , where  $\eta(0) = \eta(1) = \eta_0$ . It is a diffeomorphism of the fiber  $P$  in  $\pi: P \times O_\zeta \rightarrow O_\zeta$  obtained by parallel transport along the path  $\eta(t)$ . To compute it, consider the principal  $G_\zeta$ -bundle  $\sigma: G \rightarrow O_\zeta$ ,  $\sigma(g) = \text{Ad}_g \zeta$  and endow it with the canonical connection  $\gamma_c \in \Omega^1(G; \mathfrak{g}_\zeta)$  given by

$$\gamma_c(g) \cdot v_g = \mathbb{P}_\zeta T_g L_{g^{-1}}(v_g), \quad (12)$$

where  $v_g \in T_g G$  and  $\mathbb{P}_\zeta: \mathfrak{g} \rightarrow \mathfrak{g}_\zeta$  is the orthogonal projection relative to the bi-invariant metric  $\langle \cdot, \cdot \rangle$ . The horizontal lift of this connection is given by

$$(\text{hor}_c \lambda_{O_\zeta}(\eta))(g) = T_e R_g(P_\eta^\perp \lambda), \quad \text{for } \eta = \text{Ad}_g \zeta. \quad (13)$$

The curvature of this connection, thought of as a one-form on the base  $O_\zeta$  with values in  $\mathfrak{g}_\zeta$  (the adjoint bundle is trivial since  $G_\zeta$  is abelian), is therefore given by

$$\text{curv}(\gamma_c)(\eta)(\lambda_{O_\zeta}(\eta), \lambda'_{O_\zeta}(\eta)) = P_\zeta \text{Ad}_{g^{-1}}[P_\eta^\perp \lambda, P_\eta^\perp \lambda'] = \text{Ad}_{g^{-1}}(P_\eta[\lambda, \lambda'] - [P_\eta \lambda, P_\eta \lambda']) \quad (14)$$

where  $\eta = \text{Ad}_g \zeta$ . According to Corollary 4.2 the holonomy of the standard connection  $\gamma_c$  on  $\sigma : G \rightarrow O_\zeta$  is given by  $h_c = \exp\left(-\iint \text{curv}(\gamma_c)\right) \in G_\zeta$ , where the integral is taken over any two-manifold in  $O_\zeta$  whose boundary is given by the closed path  $\eta(t)$ . Let us rephrase this in terms of horizontal lifts of curves. By (11), if  $g(t)$  is the horizontal lift of  $\eta(t)$  in  $G$ , then it must satisfy

$$\dot{g}'(t) = T_e R_{g(t)} P_{\eta(t)}^\perp \lambda(t) \quad (15)$$

where  $\eta(t)$  is the solution of  $\dot{\eta}(t) = [\lambda(t), \eta(t)]$ , and the holonomy  $h_c \in G_\zeta$  is characterized by

$$g(1) = h_c g(0). \quad (16)$$

We shall use this to explicitly compute the holonomy of the closed path  $\eta(t)$  in  $\pi : P \times O_\zeta \rightarrow O_\zeta$ . By (9), if  $(p(t), \eta(t))$ ,  $p(0) = p_0$ ,  $h(0) = \eta_0$ , is the horizontal lift of the path  $\eta(t)$  we must have

$$\dot{p}(t) = X_{P_{\eta(t)}^\perp \lambda(t)}(p(t)), \quad p(0) = p_0.$$

But by (15),  $P_{\eta(t)}^\perp \lambda(t) = T_{g(t)} R_{g(t)^{-1}} \dot{g}(t)$ , where  $g(t)$  is the horizontal lift of  $\eta(t)$  in the canonical bundle  $\sigma : G \rightarrow O_\zeta$ . Therefore, by definition of the momentum map,  $p(t)$  satisfies the differential equation

$$\dot{p}(t) = \left( T_{g(t)} R_{g(t)^{-1}} \dot{g}(t) \right)_p (p(t)), \quad p(0) = p_0. \quad (17)$$

Now consider the curve  $g(t)g(0)^{-1} \cdot p_0$ , where  $g(t)$  is the solution of (15). Note that at  $t = 0$  this curve passes through  $p_0$ . Using the general formula  $T_g(\Phi(p))(v_g) = (T_g R_{g^{-1}}(v_g))_p(g \cdot p)$ , where  $\Phi : G \times P \rightarrow P$  is the given  $G$ -action, we see that

$$\frac{d}{dt} g(t)g(0)^{-1} \cdot p_0 = (T_{g(t)} R_{g(t)^{-1}} \dot{g}'(t))_p (g(t), g(0)^{-1} \cdot p_0) \quad (18)$$

i.e., by (17) and (18),  $p(t)$  and  $g(t)g(0)^{-1} \cdot p_0$  satisfy the same differential equation, so, by uniqueness,  $p(t) = g(t)g(0)^{-1} \cdot p_0$  and hence by (16),  $p(1) = g(1)g(0)^{-1} \cdot p_0 = h_c \cdot p_0$ . We have proved the following:

**8.2 Proposition** Let  $P$  be a Poisson manifold and  $G$  a compact Lie group acting on  $P$  with equivariant momentum map  $J : P \rightarrow \mathfrak{g}^*$ . Let  $\zeta \in \mathfrak{g}$  be a regular element and consider the torus action on  $P$  whose momentum map for each  $\eta \in O_\zeta$  is given by

$$\langle I(p, \eta), \lambda \rangle = \langle I(p), i_\eta(\lambda) \rangle, \quad \lambda \in \mathfrak{g}_\zeta$$

where  $i_\eta : \mathfrak{g}_\zeta \rightarrow \mathfrak{g}$  is given by  $i_\eta(\lambda) = \text{Ad}_h(\lambda)$  for  $h \in G_\zeta$  satisfying  $\eta = \text{Ad}_h \zeta$ . Let  $G_\zeta$  act trivially on the adjoint orbit  $O_\zeta$ . Then the trivial connection on  $\pi : P \times O_\zeta \rightarrow O_\zeta$  induces the HB connection whose Hamiltonian one-form, horizontal lift and connection form are given respectively by

$$(K \cdot [\lambda, \eta])(p, \eta) = \langle J(p), P_\eta^\perp \lambda \rangle$$

$$(\text{hor}[\lambda, \eta])(p, \eta) = \left( X_{J P_\eta^\perp(\lambda)}(p), [\lambda, \eta] \right)$$

$$\gamma(p, \eta) \cdot (v_p, [\lambda, \eta]) = v_p - X_{J P_\eta^\perp(\lambda)}(p)$$

for  $\lambda \in \mathfrak{g}$ ,  $\eta \in O_\zeta$ ,  $p \in P$ ,  $v_p \in T_p P$ . The curvature of the Hannay-Berry connection, as a two-form on the base with values in  $\mathfrak{g}_\zeta$  is given by

$$\bar{\Omega}([\lambda, \eta], [\lambda', \eta])(p, \eta) = \left( -X_{(J P_\eta^\perp(\lambda), P_\eta^\perp(\lambda'))}(p), 0 \right).$$

The Hannay-Berry connection preserves the level sets of  $I$ . The holonomy of the closed path  $\eta(t)$  in  $O_\zeta$  is the diffeomorphism of  $P$  given by the action of the group element  $h_c$  representing the holonomy of  $\eta(t)$  in the canonical bundle  $\sigma : G \rightarrow O_\zeta$ .

**Remark** When the  $G_\zeta$ -action is free, there is an alternative description of the bundle and the HB connection on  $I^{-1}(\mu)$  within  $P \times O_\zeta$ . Consider the two projections

$$O_\zeta \xleftarrow{p_1} I^{-1}(\mu) \xrightarrow{p_2} I^{-1}(\mu)/G.$$

By equivariance,  $p_1(p_2^{-1}(x)) = O_\zeta$  for each  $x \in I^{-1}(\mu)/G$ . By freeness,  $p_2^{-1}(x) \cong G$ . Thus, restricting  $p_1$  to  $p_2^{-1}(x)$ , we have the homogeneous bundle  $G \rightarrow O_\zeta = G/G_\zeta$ . Consequently,

$$I^{-1}(\mu) = \bigcup_{x \in I^{-1}(\mu)} p_2^{-1}(x)$$

is a bundle of homogeneous bundles. In the case of the Foucault pendulum,  $I^{-1}(\mu)/G$  is a point, so that  $I^{-1}(\mu) \cong G = \mathbf{SO}(3)$ .

## §9 The Hannay-Berry Connection on Level Sets of the Momentum Map

By property **a** of Theorem 7.1, the HB connection induces a connection on any level set of  $I$ . Indeed, if  $\mu \in \mathfrak{g}^*$  is a (weakly) regular value, then  $T_p(I^{-1}(\mu)) = \ker dI(p)$ , and property **a** states that  $dI(p)$  vanishes on horizontal vectors, i.e. the Hannay-Berry horizontal lift at a point of  $I^{-1}(\mu)$  is necessarily tangent to  $I^{-1}(\mu)$ . The proof of Theorem 7.1 i shows that this connection is unique with the properties **b** and **c**.

**9.1 Proposition** *The HB connection induces a unique connection on each submanifold  $I^{-1}(\mu)$  (for  $\mu$  a weakly regular value) satisfying properties **b** and **c** of Theorem 7.1.*

There is another way to describe this connection. Fix a weakly regular value  $\mu \in \mathfrak{g}^*$  of  $I$  and let  $\pi_\mu : I^{-1}(\mu) \rightarrow M$  be the restriction of  $\pi$  to  $I^{-1}(\mu)$ . Let us assume that  $\pi_\mu$  is surjective. This is in general not the case. If  $G = \mathbb{T}^n$ , Golin et al. [1989] prove that  $\pi_\mu$  is surjective; see also Proposition 10.6 where we sketch their proof. Let us assume throughout this section that  $\pi_\mu$  is surjective. It is also a submersion:  $T\pi$  restricted to any horizontal subspace is an isomorphism onto the corresponding tangent space to  $M$  and, as we just argued,  $\ker dI(p)$  contains the horizontal space at  $p$ . The group  $G$  does not act on  $I^{-1}(\mu)$ , but the coadjoint isotropy subgroup  $G_\mu$  does. For  $Z \in \mathfrak{X}(M)$ , define

$$\text{hor}_\mu Z = \langle \text{hor}_0 Z \rangle_\mu := \frac{1}{|G_\mu|} \int_{G_\mu} \Phi_g^*(\text{hor}_0 Z) dg \quad (1)$$

where  $|G_\mu|$  is the volume of  $G_\mu$  in the induced Haar measure  $dg$  on  $G_\mu$  from the one given on  $G$ . As before, we assume that  $\pi : E \rightarrow M$  is endowed with a Poisson-Ehresmann connection  $\text{hor}_0$ . *A priori*, it is not clear that  $\text{hor}_\mu Z$  is a vector field tangent to  $I^{-1}(\mu)$ . To show this, let  $p \in I^{-1}(\mu)$ ,  $m = \pi(p)$ ,  $v \in T_m M$ , and  $\xi \in \mathfrak{g}_\mu$ , the Lie algebra at  $G_\mu$ . By equivariance of  $I$  (if it is not equivariant, work with the induced affine action of  $G$  on  $\mathfrak{g}^*$  — see Abraham and Marsden [1978, Proposition 4.2.7] for this procedure), we have

$$\begin{aligned} I^{g^{-1}\xi}(p) &= I(p) (\text{Ad}_{g^{-1}} I(p)) \cdot \xi = \text{Ad}_g^* \mu \cdot \xi = \mu \cdot \xi \\ &= I(p) (\xi) = I^\xi(p) = I^\xi(g \cdot p) = (\Phi_g^* I^\xi)(p), \end{aligned}$$

so that

$$dI^\xi(p) \cdot (\text{hor}_\mu v)(p) = \frac{1}{|G_\mu|} \int_{G_\mu} dI^\xi \cdot \Phi_g^*(\text{hor}_0 v)(p) dg$$

$$\begin{aligned}
&= \frac{1}{|G_\mu|} \int_{G_\mu} dI_{\mathfrak{g}^{-1} \cdot \xi}(p) \cdot \Phi_g^*(\text{hor}_0 v)(p) dg \\
&= \frac{1}{|G_\mu|} \int_{G_\mu} \Phi_g^*(D_0 I^\xi \cdot \text{hor}_\mu v)(p) dg \\
&= \frac{1}{|G_\mu|} \int_{G_\mu} (\Phi_g^*(D_0 I^\xi) \cdot \text{hor}_\mu v)(p) dg = \langle D_0 I^\xi \rangle_\mu(p) \cdot (\text{hor}_\mu v)(p) .
\end{aligned}$$

Therefore, denoting by  $D_\mu$  the covariant derivative induced by  $\text{hor}_\mu$ , we get the analog of Proposition 6.3 vii,

$$D_\mu I = \langle D_0 I \rangle_\mu . \quad (2)$$

Thus  $\langle D_0 I \rangle_\mu = 0$ , iff  $D_\mu I = 0$  and  $\text{hor}_\mu v$  is tangent to  $I^{-1}(\mu)$ .

The verification of properties **b** and **c** is done as in Proposition 6.3 with  $G$  replaced by  $G_\mu$ ,  $E$  by  $I^{-1}(\mu)$ , and  $G$ -averages by  $G_\mu$ -averages. Therefore by Proposition 9.1, (1) defines the unique connection on  $I^{-1}(\mu) \rightarrow M$  satisfying properties **b** and **c** of Theorem 7.1, since if  $\langle K \cdot Z \rangle_\mu = 0$ , by Fubini's theorem on bundles, it follows that  $\langle K \cdot Z \rangle = 0$ . We have proved the following:

**9.2 Theorem** *In the hypotheses of Theorem 7.1, assume in addition that  $I^{-1}(\mu) \rightarrow M$  is onto for all weakly regular values  $\mu$  and  $\langle D_0 I \rangle_\mu = 0$  for all  $\mu \in \mathfrak{g}^*$ . Then (1) defines the induced HB connection on  $I^{-1}(\mu) \rightarrow M$  for all weakly regular values  $\mu$  of  $I$ .*

**Remark** If  $I^{-1}(\mu) \rightarrow M$  is a principal  $G_\mu$ -bundle, then the HB connection is a principal connection, since the horizontal lift is manifestly  $G_\mu$ -invariant. Such a situation occurs for slowly varying integrable systems discussed in the next section.

## §10 Case I: Bundles with the Canonical Connection; Integrable Systems and Hannay's Angle

In this section we treat a large class of Poisson fiber bundles which come equipped with a Poisson-Ehresmann connection and study the induced HB connection.

A *bundle of symplectic manifolds* is a fiber bundle  $\pi : E \rightarrow M$  all of whose fibers are symplectic and whose transition functions are symplectic. Gotay et al. [1980] gave conditions guaranteeing the existence of a presymplectic (i.e., closed) form  $\omega$  on  $E$  whose pull-back to each fiber is the given fiber symplectic form. We assume that such a *closed* two-form  $\omega$  on  $E$  is given and call  $\pi : (E, \omega) \rightarrow M$  a *coherent bundle of symplectic manifolds*.

As remarked in Gotay et al. [1983], *any coherent bundle of symplectic manifolds comes equipped with an Ehresmann connection*. Indeed, if  $V = \ker T\pi$  is the vertical bundle of  $TE$ , then each fiber  $V_p$  of  $V$ , where  $p \in E$ , is a symplectic vector space. Define the subbundle  $H$  to be the  $\omega$ -orthogonal complement to  $V$

$$H_p = V_p^\omega = \{u \in T_p E \mid \omega(p)(u, v) = 0 \text{ for all } v \in V\};$$

$H = \bigcup_{p \in E} H_p$  is a subbundle of  $TE$  since the rank of  $\omega$  is constant on connected components of  $M$ . We claim that  $TE = H \oplus V$ . To prove this, note that if  $u \in H \cap V$  then  $\omega(u, v) = 0$  for all  $v \in V$ , so that  $u = 0$  by non-degeneracy of  $\omega$  on  $V$  and thus  $H \cap V = \{0\}$ . Now consider the map  $T_p E \rightarrow T_p^* E \rightarrow V_p^*$ , the first arrow being  $u \mapsto \omega(u, \cdot)$  and the second, the restriction of a linear functional on  $T_p E$  on the subspace  $V_p$ . Since  $V_p$  is symplectic, this linear map  $T_p E \rightarrow V_p^*$  is onto. The kernel of this map equals  $V_p^\omega = H_p$  and thus  $\dim T_p E - \dim H_p = \dim V_p^* = \dim V_p$ . This proves the claim. Thus,  $H$  is the horizontal subbundle of an Ehresmann connection on  $E$ . Let  $\text{hor}_0$  denote its horizontal lift:  $(\text{hor}_0 w)(p) = (T_p \pi)^{-1} w$  for  $w \in T_{\pi(p)} M$ . We denote by  $D_0$  the covariant differentiation defined by  $\text{hor}_0$ .

A function  $f : E \rightarrow \mathbb{R}$ , defines a *vertical Hamiltonian vector field*  $X_f$  by

$$j_m^* df = i_{X_f} j_m^* \omega, \quad (1)$$

where  $\pi(p) = m$  and  $j_m$  is the fiber inclusion  $\pi^{-1}(m) \rightarrow E$ . We note that this is not the usual Hamiltonian vector field corresponding to the presymplectic form  $\omega$ . The *vertical Poisson bracket* is defined by  $\{f, h\} = \omega(X_f, X_h)$ . One easily checks that:

**10.1 Proposition** *A projectable vector field  $Y \in \mathfrak{X}(E)$  is a Poisson bracket derivation if and only if the Lie derivative  $\mathfrak{L}_Y \omega$  vanishes on any pair of vertical vectors.*



Now let us assume that  $Y = \text{hor}_0 Z$  for some  $Z \in \mathfrak{X}(M)$ . By Proposition 10.1,  $\text{hor}_0 Z$  is a Poisson bracket derivation if and only if  $(\mathfrak{L}_{\text{hor}_0 Z} \omega)(X_f, X_h) = 0$  for any  $f, h : E \rightarrow \mathbb{R}$ . Since  $\omega$  is closed we have

$$\begin{aligned} (\mathfrak{L}_{\text{hor}_0 Z} \omega)(X_f, X_h) &= (d\mathfrak{L}_{\text{hor}_0 Z} \omega)(X_f, X_h) \\ &= X_f[\omega(\text{hor}_0 Z, X_h)] - X_h[\omega(\text{hor}_0 Z, X_f)] - \omega(\text{hor}_0 Z, [X_f, X_h]) = 0 \end{aligned} \quad (2)$$

since  $\omega$  vanishes on every pair of horizontal and vertical vector fields; the last term is also zero since  $[X_f, X_h] = -X_{\{f, h\}}$  is vertical. Thus we have proved the following:

**10.2 Proposition** *The canonical connection  $\text{hor}_0$  of the coherent bundle of symplectic manifolds  $\pi : (E, \omega) \rightarrow M$  is a Poisson-Ehresmann connection.*

**10.3 Proposition** *Let  $\Omega_0$  denote the curvature of the canonical connection  $\text{hor}_0$ . Then for any  $Z_1, Z_2 \in \mathfrak{X}(M)$  we have*

$$\bar{\Omega}_0(Z_1, Z_2) = -\omega(\text{hor}_0 Z_1, \text{hor}_0 Z_2) . \quad (3)$$

**Proof** For  $Z_1, Z_2 \in \mathfrak{X}(M)$  and  $V \in \mathfrak{X}_{\text{vert}}(E; M)$ ,

$$\begin{aligned} (i_{\bar{\Omega}_0(Z_1, Z_2)} \omega)(V) &= \omega(\bar{\Omega}_0(Z_1, Z_2), V) \\ &= \omega(\text{hor}_0[Z_1, Z_2], V) - \omega([\text{hor}_0 Z_1, \text{hor}_0 Z_2], V) \\ &= -\omega([\text{hor}_0 Z_1, \text{hor}_0 Z_2], V) . \end{aligned}$$

Now we use the formula:

$$\begin{aligned} (d\omega)(A, B, C) &= A[\omega(B, C)] - B[\omega(A, C)] + C[\omega(A, B)] \\ &\quad - \omega([A, B], C) + \omega([A, C], B) - \omega([B, C], A) \end{aligned} \quad (4)$$

with  $A = \text{hor}_0 Z_1$ ,  $B = \text{hor}_0 Z_2$ ,  $C = V$ . In (4), the first and second summands vanish. Also, since the bracket of a vertical with a horizontally lifted vector field is again vertical, the fifth and sixth terms also vanish. Since  $d\omega = 0$  by hypothesis, we get

$$V[\omega(\text{hor}_0 Z_1, \text{hor}_0 Z_2)] = \omega([\text{hor}_0 Z_1, \text{hor}_0 Z_2], V) ,$$

i.e.,

$$(i_{\bar{\Omega}_0(Z_1, Z_2)} \omega)(V) = d(\omega(\text{hor}_0 Z_1, \text{hor}_0 Z_2)) \cdot V . \blacksquare$$

If  $G$  is a compact Lie group defining a family of Hamiltonian  $G$ -actions on  $\pi : (E, \omega) \rightarrow M$ , then the action  $\Phi$  restricted to every fiber  $\pi^{-1}(m)$  preserves  $\omega$  pulled back to  $\pi^{-1}(m)$ . If we denote by  $\Phi^m : G \times \pi^{-1}(m) \rightarrow \pi^{-1}(m)$  the restriction of the  $G$ -action  $\Phi$  to  $\pi^{-1}(m)$ , then this

says that  $(\Phi_g^m)^*\omega = \omega$  on  $\pi^{-1}(m)$ . In general,  $\Phi_g^*\omega \neq \omega$  on  $E$ . Let  $\text{hor}_0$  denote the horizontal lift of the connection induced by  $\omega$ . Let  $\text{hor}$  denote the horizontal lift of the induced Hannay-Berry connection.

**10.4 Theorem** Assume that there is an equivariant momentum map  $I$  in the presymplectic sense for the averaged form  $\langle\omega\rangle$  on  $E$ , i.e.,

$$dI^\xi = i_{\xi_E}\langle\omega\rangle \quad (5)$$

for all  $\xi \in \mathfrak{g}$ . Then the following hold.

- i  $I: E \rightarrow \mathfrak{g}^*$  is a parametrized momentum map for the family of Hamiltonian  $G$ -actions on  $E$ .
- ii There is a unique decomposition  $\langle\omega\rangle = \omega + d\sigma$ , where  $\sigma$  is a one-form on  $E$  annihilating all vertical vectors.
- iii The Hamiltonian one-form  $Z \mapsto K \cdot Z$  of the HB connection is  $\sigma \circ \text{hor}_0$ .
- iv The horizontal distribution of the HB connection is the  $\langle\omega\rangle$ -orthogonal complement of the vertical subbundle.
- v Let  $\tilde{\Omega}(X, Y) = -\langle\omega\rangle(X^h, Y^h)$ , where  $X, Y \in \mathfrak{X}(E)$  and  $X^h, Y^h$  denote their horizontal parts relative to  $\text{hor}$ . The horizontal distribution of the HB connection equals the characteristic subbundle of  $\langle\omega\rangle + \tilde{\Omega}$ , i.e. a vector  $v \in T_p E$  is horizontal if and only if  $\langle\omega\rangle(v, u) + \tilde{\Omega}(v, u) = 0$  for all  $u \in T_p E$ .
- vi The HB connection preserves the level sets of  $I$ , i.e.,  $DI = 0$ .
- vii The adiabatic condition  $\langle D_0 I \rangle = 0$  holds.
- viii The curvature of the HB connection is a Hamiltonian vector field with Hamiltonian function equal to  $\tilde{\Omega}(\text{hor } Z_1, \text{hor } Z_2)$  for  $Z_1, Z_2 \in \mathfrak{X}(M)$ , i.e.,

$$\bar{\Omega}(Z_1, Z_2) = X_{-\langle\omega\rangle(\text{hor } Z_1, \text{hor } Z_2)}.$$

**Remarks 1** Properties ii, iv, v, and vi were obtained for the case of a trivial bundle with fiber an exact symplectic manifold and  $\text{hor}_0$  the trivial connection by Golin, Knauf, and Marmi [1989].

**2** If the  $G$ -action preserves  $\omega$ , then  $D = D_0$  for then  $\omega = \langle\omega\rangle$ .

**Proof** i We begin by showing that  $\omega$  and  $\langle\omega\rangle$  coincide on vertical vectors. Indeed, if  $u, v \in V_p = \ker T_p \pi$  are vertical vectors, then  $T_p \Phi_g^m(u) = T_p \Phi_g^m(u)$  and similarly for  $v$ , so that

$$\langle\omega\rangle(p)(u, v) = \frac{1}{|G|} \int_G (\Phi_g^*\omega)(p)(u, v) dg = \frac{1}{|G|} \int_G \omega(\Phi_g^m(p))(T_p \Phi_g^m(u), T_p \Phi_g^m(v)) dg$$

$$= \frac{1}{|G|} \int_G ((\Phi_g^m)^* \omega)(p)(u, v) dg = \omega(p)(u, v) .$$

Thus, if  $I$  is the momentum map defined by  $\langle \omega \rangle$  on  $E$  (in the presymplectic sense), then for any  $\xi \in \mathfrak{g}$ ,  $v \in V_p$  we have

$$dI^\xi(p)(v) = \langle \omega \rangle(p)(\xi_E(p), v) = \omega(p)(\xi_E(p), v)$$

since both  $\xi_E(p)$  and  $v$  are vertical. But this says that on the fiber through  $p$ ,  $I^\xi$  is a momentum map of the  $G$ -action, i.e.  $I$  is a parametrized momentum map of the action.

II Since  $G$  is compact, the exponential map is onto. So if  $g \in G$ , write  $g = \exp \xi$  and hence

$$\begin{aligned} \Phi_g^* \omega - \omega &= \int_0^1 \frac{d}{dt} \Phi_{\exp t\xi}^* \omega dt = \int_0^1 \Phi_{\exp t\xi}^* \mathcal{L}_{\xi_E} \omega dt = \int_0^1 \Phi_{\exp t\xi}^* d i_{\xi_E} \omega dt \\ &= d \left[ \int_0^1 \Phi_{\exp t\xi}^* i_{\xi_E} (\omega - \langle \omega \rangle) dt + \int_0^1 \Phi_{\exp t\xi}^* i_{\xi_E} \langle \omega \rangle dt \right] = d\sigma_g , \end{aligned}$$

since

$$\int_0^1 \Phi_{\exp t\xi}^* i_{\xi_E} \langle \omega \rangle dt = \int_0^1 \Phi_{\exp t\xi}^* dI^\xi dt = dI^\xi ,$$

where we have written

$$\sigma_g = \int_0^1 \Phi_{\exp t\xi}^* i_{\xi_E} (\omega - \langle \omega \rangle) dt \in \Omega^1(E) . \quad (6)$$

Since  $\omega$  and  $\langle \omega \rangle$  coincide on vertical vector fields,  $\sigma_g$  annihilates all vertical vectors; hence  $\sigma = \langle \sigma \rangle$  will also annihilate all vertical vectors. Averaging we obtain:  $\langle \omega \rangle - \omega = d\sigma$ .

III In the proof of Proposition 6.3, we gave an explicit formula for  $K \cdot Z$ , namely

$$K \cdot Z = \langle f_Z \rangle, \quad (7)$$

where

$$f_Z(g) = - \int_0^1 \Phi_{\exp t\xi}^* (dI^\xi \cdot \text{hor}_0 Z) dt \quad (8)$$

for  $g = \exp \xi$ . However, since

$$i_{\xi_E} (\omega - \langle \omega \rangle) \cdot \text{hor}_0 Z = - i_{\xi_E} \langle \omega \rangle (\text{hor}_0 Z) = - dI^\xi \cdot \text{hor}_0 Z ,$$

we see that (6) and (8) imply that  $K \cdot Z = \sigma(\text{hor}_0 Z)$ .

**iv** We have seen in **i** that the presymplectic form  $\langle \omega \rangle$  when restricted to each fiber coincides with  $\omega$ , and hence is symplectic in each fiber. Therefore,  $(E, \langle \omega \rangle)$  is a coherent bundle of symplectic manifolds and hence defines a canonical connection whose horizontal subbundle is  $V^{(\omega)}$ , the  $\langle \omega \rangle$ -orthogonal complement of  $V$ . On the other hand, if  $Y$  is a vertical vector field, by **iii**, the fact that  $\sigma(W) = 0$  for any vertical vector field  $W$ , and  $\text{hor } Z = \text{hor}_0 Z + X_{K \cdot Z}$ , we get

$$\begin{aligned} (i_{\text{hor } Z} \langle \omega \rangle)(Y) &= \langle \omega \rangle(\text{hor } Z, Y) = \omega(\text{hor}_0 Z + X_{\sigma(\text{hor}_0 Z)}, Y) + d\sigma(\text{hor } Z, Y) \\ &= \omega(X_{\sigma(\text{hor}_0 Z)}, Y) + (\text{hor } Z)[\sigma(Y)] - Y[\sigma(\text{hor } Z)] - \sigma([\text{hor } Z, Y]) \\ &= d(\sigma(\text{hor}_0 Z)) \cdot Y - Y[\sigma(\text{hor}_0 Z + X_{K \cdot Z})] \\ &= Y[\sigma(\text{hor}_0 Z)] - Y[\sigma(\text{hor}_0 Z)] = 0, \end{aligned}$$

which says that the horizontal subbundle of the HB connection is included in  $V^{(\omega)}$ . Since its complement is  $V$ , it equals  $V^{(\omega)}$ .

**v** Let  $u, v \in T_p E$  and write  $u = u_h + u_v, v = v_h + v_v$ . Using **iv**, write

$$\langle \omega \rangle(u, v) - \tilde{\Omega}(u, v) = \omega(u_v, v_v)$$

which shows that the left hand side vanishes for all  $u \in T_p E$  if and only if  $v_v = 0$ , i.e.,  $v$  is horizontal.

**vi** The statement is equivalent to

$$dI^\xi \cdot \text{hor } Z = 0$$

for all  $\xi \in \mathfrak{g}$ ,  $Z \in \mathfrak{X}(M)$ . By **vi** and the definition of **I** we get

$$dI^\xi \cdot \text{hor } Z = \langle \omega \rangle(\xi_E, \text{hor } Z) = 0.$$

**vii** By Proposition 6.3 **vii**,  $DI = 0$  is equivalent to  $\langle D_0 I \rangle = 0$ . The result now follows from **vi**.

**viii** The curvature formula is a direct consequence of Theorem 7.1, Corollary 7.4, and Proposition 10.3 provided we show

$$\langle \omega \rangle(\text{hor } Z_1, \text{hor } Z_2) = \langle \{K \cdot Z_1, K \cdot Z_2\} \rangle + \langle \omega(\text{hor}_0 Z_1, \text{hor}_0 Z_2) \rangle.$$

By  $G$ -invariance of  $\text{hor } Z_1$  and  $\text{hor } Z_2$  we have

$$\begin{aligned} \langle \omega \rangle(\text{hor } Z_1, \text{hor } Z_2) &= \frac{1}{|G|} \int_G \Phi_g^*(\omega(\text{hor } Z_1, \text{hor } Z_2)) dg \\ &= \frac{1}{|G|} \int_G \Phi_g^*\{K \cdot Z_1, K \cdot Z_2\} dg + \frac{1}{|G|} \int_G \Phi_g^*(\omega(\text{hor}_0 Z_1, \text{hor}_0 Z_2)) dg \end{aligned}$$

$$= \langle \{K \cdot Z_1, K \cdot Z_2\} \rangle + \langle \omega(\text{hor}_0 Z_1, \text{hor}_0 Z_2) \rangle . \blacksquare$$

There are important cases in which the hypothesis of this theorem is always satisfied.

**10.5 Proposition** *Assume that the coherent bundle of symplectic manifolds  $\pi : (E, \omega) \rightarrow M$  is exact, i.e., there is a one-form  $\theta$  on  $E$  such that  $\omega = -d\theta$ . Then any family of symplectic  $G$ -actions is Hamiltonian and its parametrized momentum map is given by*

$$I(p) \cdot \xi = \langle \theta \rangle (\xi_E) .$$

**Proof**  $\langle \theta \rangle$  is  $G$ -invariant and hence  $\mathfrak{L}_{\xi_E} \langle \theta \rangle = 0$ . This says that

$$i_{\xi_E} \langle \omega \rangle = -i_{\xi_E} d\langle \theta \rangle = d i_{\xi_E} \langle \theta \rangle ,$$

$$\text{i.e., } i_{\xi_E} \langle \theta \rangle = I^\xi . \blacksquare$$

The situation described in this section for  $E = P \times M$ , where  $(P, \omega)$  is a given symplectic manifold,  $\pi : P \times M \rightarrow M$  the canonical projection, using the presymplectic form on  $E$  obtained by pulling back  $\omega$  by the projection of  $E$  to  $P$ , and  $\text{hor}_0$  the trivial connection, has been treated in detail by Montgomery [1988] and by Golin, Knauf, and Marmi [1989]. The following proposition appears in these papers.

**10.6 Proposition** *Suppose that the parameter dependent Hamiltonian  $H : P \times M \rightarrow \mathbb{R}$  defines a completely integrable system for each value of the parameter  $m \in M$  and that  $I : P \times M \rightarrow \mathbb{R}^n$  is a parametrized set of global action variables,  $\dim P = 2n$ . Assume that the adiabatic condition  $\langle d_M I \rangle = 0$  holds. Then the HB connection on  $P \times M$  preserves the level sets of  $I$ . If  $\mu \in \mathbb{R}^n$  is a regular value of  $I(\cdot, m_0)$  for a fixed  $m_0 \in M$ , if  $I^{-1}(\mu)$ ,  $I^{-1}(\mu) \cap (P \times \{m\})$  and  $M$  are connected, and if the sets  $I^{-1}(\mu) \cap (P \times \{m\})$  are compact for all  $m \in M$ , then  $I^{-1}(\mu) \rightarrow M$  is a principal torus bundle and the HB connection is a principal connection.*

*For loops  $c$  in  $M$  based at  $m_0$  (small enough in general, or arbitrary, if we guarantee that  $I^{-1}(\mu) \rightarrow M$  is a principal torus bundle), the holonomy of the HB connection coincides with Hannay's angle [1985], namely*

$$\Delta \theta = \int_c \langle d_M \theta \rangle ,$$

where  $(\theta_1, \dots, \theta_n)$  is the parameter dependent set of angle variables defined by  $I$ .

**Proof** Assume for the moment (as in §9) that the restriction of  $\pi$  to  $I^{-1}(\mu)$  covers  $M$ . We first prove the theorem under this additional hypothesis. The first part is a direct consequence of the general theory if we prove that  $I^{-1}(\mu) \rightarrow M$  is a principal torus bundle. Since  $I^{-1}(\mu)$  is compact, parallel translation in  $I^{-1}(\mu)$  is complete. Parallel translation along a path connecting  $m_0$  to an arbitrary  $m \in M$  defines a diffeomorphism of the fiber  $I^{-1}(\mu) \cap (P \times \{m_0\})$  with the fiber  $I^{-1}(\mu) \cap (P \times \{m\})$ . Both fibers are orbits of the torus action on  $P \times M$  defined by the integrability assumption of  $H$  and this diffeomorphism intertwines the torus action. Thus  $I^{-1}(\mu) \rightarrow M$  is a principal torus bundle.

Next we compute the horizontal lift of the HB connection induced by the trivial connection  $(\text{hor}_0 Z)(p, m) = (0, Z(m))$ . Since  $\text{hor}$  is the horizontal lift for the averaged connection,

$$(\text{hor } Z)[f](p) = \frac{1}{|G|} \int_G (\text{hor}_0 Z)[f \circ \Phi_{g^{-1}}](g \cdot p) dg \quad (9)$$

for  $Z \in \mathfrak{X}(M)$ ,  $f: E \rightarrow \mathbb{R}$ ,  $p \in E$ , and  $g \in G$ . Apply (9) to our case taking  $f$  to be each of  $\theta^i$ ,  $I^i, x^a$  and  $Z = \partial/\partial x^a$ , where  $I^i$  are the coordinate representations of the action  $I: P \times M \rightarrow \mathfrak{g}^*$  for each fixed value of the parameter, and  $\theta^i$  are the corresponding canonically conjugate variables, i.e., the angle variables defined by  $I$ . We get

$$\left( \text{hor } \frac{\partial}{\partial x^a} \right) [I^i] = \left\langle \frac{\partial I^i}{\partial x^a} \right\rangle, \quad \left( \text{hor } \frac{\partial}{\partial x^a} \right) [\theta^i] = \left\langle \frac{\partial \theta^i}{\partial x^a} \right\rangle, \quad \text{and} \quad \left( \text{hor } \frac{\partial}{\partial x^a} \right) [x^b] = \delta^{ab},$$

i.e.,

$$\text{hor } \frac{\partial}{\partial x^a} = \left\langle \frac{\partial I^i}{\partial x^a} \right\rangle \frac{\partial}{\partial I^i} + \left\langle \frac{\partial \theta^i}{\partial x^a} \right\rangle \frac{\partial \theta^i}{\partial x^a} \frac{\partial}{\partial \theta^i} + \frac{\partial}{\partial x^a}$$

By hypothesis  $\langle \partial I^i / \partial x^a \rangle = 0$ , so that

$$\text{hor } \frac{\partial}{\partial x^a} = \left\langle \frac{\partial \theta^i}{\partial x^a} \right\rangle \frac{\partial \theta^i}{\partial x^a} \frac{\partial}{\partial \theta^i} + \frac{\partial}{\partial x^a}.$$

Therefore, if  $c$  is a curve in  $M$  and if  $(\theta^i(t), I^i(t), x^a(t))$  is the horizontal lift of  $c$  given in coordinates by  $(x^a(t))$ ,  $t \in [0, 1]$ , then the differential equations defining the horizontal lift are

$$\frac{d\theta^i}{dt} = \left\langle \frac{\partial \theta^i}{\partial x^a} \right\rangle \frac{dx^a}{dt}, \quad \frac{dI^i}{dt} = 0.$$

Integrate the first equation over the path  $c$  with initial condition  $\theta^i(0) = \theta_0^i$  and get

$$\theta(1) = \theta_0 + \int_c \langle d_M \theta \rangle.$$

Thus, if  $c$  is a loop based at  $m_0$ , the holonomy of this path is  $\theta(1) - \theta_0 = \int_c \langle d_M \theta \rangle$ , which is the formula for Hannay's angles.

To complete the proof of the theorem, we address the issue of why  $I^{-1}(\mu)/T^n$  is diffeomorphic to the parameter space  $M$ . The projection  $\pi : P \times M \rightarrow M$  induces a map  $I^{-1}(\mu)/T^n \rightarrow M$  which is smooth and injective since  $I^{-1}(\mu) \cap (P \times \{m\})$  consists of exactly one torus, as we have seen at the beginning of the proof, using parallel translation. This argument also shows that it is a local diffeomorphism, so all that remains to be shown is that it is surjective. This is equivalent to proving that the set of values of  $I$  is parameter independent. We begin by remarking that by the generalization of Poincaré's last geometric theorem, the tori for nearby values of the parameters intersect (see Arnold [1978], Appendix 3). To prove the theorem, we need the *global* version of this result, for our case, i.e., if as assumed,  $\mu$  is a regular value of  $I(\cdot, m_0) =: I_{m_0}$  then we need to show that for *all*  $m \in M$  the tori  $I_{m_0}^{-1}(\mu)$  and  $I_m^{-1}(\mu)$  intersect. If this is shown, then clearly  $\mu$  is in the range of  $I_m$  and thus the range of the parametrized momentum map is parameter independent.

An ingenious proof of this fact was given by Golin, Knauf, and Marmi [1989]. We reproduce here their proof for the sake of completeness. Let  $c(t)$ ,  $t \in [0, 1]$  be a smooth path in  $M$  connecting  $m_0$  to  $m$ . Then the Hamiltonian one-form of the HB connection given by  $c'(t)$ , namely  $K \cdot c'(t) = \sigma(c'(t))$  defines a time dependent Hamiltonian on  $P$ . Its evolution operator (flow) gives hence a parameter dependent symplectic transformation  $\psi_t : P \rightarrow P$ . It is this transformation, which by horizontal lift, maps the tori of different values of the parameters to each other, as we saw at the beginning of the proof. So, what one has to exclude is the situation that an orbit starting at a point on the torus  $I_{m_0}^{-1}(\mu)$  is undefined for a time  $t_0 < 1$ .

Let us argue by contradiction and assume that such an orbit and  $t_0$  do exist. Then there is a smaller  $t_1 < t_0$  such that the torus  $I_{c(t_1)}^{-1}(\mu)$  does not intersect  $I_{m_0}^{-1}(\mu)$  for otherwise, by compactness there would be an accumulation point  $p \in I_{m_0}^{-1}(\mu)$  with  $I(p, c(t_1)) = \mu$ . By openness of regular values, there is an open neighborhood  $U$  of  $\mu$  and  $t_2 \in ]t_1, t_0[$  such that  $I_{c(t)}^{-1}(v) \neq \emptyset$  for all  $v \in U$  and  $t \in [0, t_2]$ . By Jost's theorem,  $P$  fibers locally as a torus bundle, i.e., locally  $P$  is of the form  $B \times T^n$ , for  $B$  an open ball in  $\mathbb{R}^n$ . By the Lagrangian embedding theorem, there is an embedding  $\lambda$  of this piece of  $P$  into  $T^*T^n$ , which sends the torus  $I_{m_0}^{-1}(\mu)$  to the zero section. Since the set  $S = \{p \in P \mid I_{c(t)}^{-1}(p) \in U \text{ for some } t \in [0, t_2]\}$  is compact with non-empty interior, define a time dependent Hamiltonian  $H_t$  on  $T^*T^n$  with  $t$ -independent compact support  $\lambda(S)$  by  $H_t(\lambda(p)) = \sigma(c'(t))(p)$  for all  $p$  in some open subset of  $S$  given by shrinking  $U$  to a smaller neighborhood of  $\mu$ . Let  $\phi_t$  be the time dependent flow in  $T^*T^n$  generated by  $H_t$ . So  $\phi_t$  is a family of Hamiltonian isotopies. The other tori  $I_{c(t)}^{-1}(\mu)$  are Lagrangian submanifolds of  $T^*T^n$  via the embedding  $\lambda$ . So, we reduced the problem to the study of the intersection of the Lagrangian manifolds  $T^n$  and  $\phi_t(T^n)$  in  $T^*T^n$ . Hofer [1985] has shown that

the number of intersections of  $\phi_1(N)$  with  $N$  in  $T^*N$  for  $N$  compact is at least  $CL(N) + 1$ , where  $CL(N)$  is the cup-length of the homology ring of  $N$ ; for generalizations see Laudenchbach and Sikorav [1985]. This result implies for our case that  $I_{m_0}^{-1}(\mu) \cap I_{c(t_2)}^{-1}(\mu)$  contains at least  $n + 1$  points, a contradiction. ■

## Remarks

1 For the non-integrable case, we suspect that the image of the momentum map can be parameter dependent.

2 Connectivity of the fibers  $I^{-1}(\mu) \cap (P \times \{m\})$  is automatic if  $P$  is compact. This was proved by Atiyah [1982] and Guillemin and Sternberg [1984] in the course of the proof of the convexity theorem for images of momentum maps given by toral actions. Moreover, the image of  $I(\cdot, m)$  up to translation, depends only on the cohomology class of  $P$ . (V. Guillemin, private conversation.) We can fix the image by insisting that its center of mass is the origin:  $\int_P I(\cdot, m) = 0$ . Since the image is independent of the parameter,  $I^{-1}(\mu)$  automatically projects onto all of  $M$ .

3 The adiabatic condition  $\langle d_M I \rangle = 0$  is not always satisfied. For example, if  $I$  defines global action variables on  $P$  and  $f: M \rightarrow \mathbb{R}^n$ , then  $I + f \circ \pi$  generates the same toral action as  $I$ , but  $\langle d_M(I + f \circ \pi) \rangle = \langle d_M I \rangle + d_M f$ .

4 If we deal with slowly varying one degree of freedom systems,  $\langle d_M I \rangle = 0$  implies that  $I$  is an adiabatic invariant (Arnold [1978], [1983]). This means that for a time scale of order  $1/\epsilon$ , the actions vary by  $O(\epsilon)$  as  $\epsilon \rightarrow 0$ ;  $\epsilon$  measures how slowly the system varies.\* The proof of adiabatic invariance depends on the averaging method, and fails for higher degree of freedom systems. Instead, an almost adiabatic invariance holds. This means that for all initial conditions *except* for a set of measure  $O(\epsilon)$ , adiabatic invariance holds. See Golin and Marmi [1989] or Arnold [1978] and especially references to Neistadt therein. ♦

Let us address this last question in more detail. By the Liouville-Arnold and Jost theorems for completely integrable systems whose level sets for the action are compact, we have generically a locally trivial-fibration of phase space as a torus bundle. Now consider a parameter dependent family of Hamiltonian systems with parametrized actions  $I: P \times M \rightarrow \mathbb{R}^n$ . Restrict  $I$  to the open subset of  $P$  where  $P$  is  $B \times T^n$ , for  $B$  an open ball in  $\mathbb{R}^n$ . Consider a basis  $(c_1, \dots, c_n)$  of the first homology of  $T^n$  ( $T^n$  thought of as being the torus at the point  $(p_0, m_0)$ ), and choose them to depend smoothly by on  $(p, m)$ , in a neighborhood of  $(p_0, m_0)$ . Let  $\theta$  be a one form on the neighborhood of  $(p_0, m_0)$  such that  $\omega = -d\theta$ , i.e., in the identification of this neighborhood

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\* It is a classical result of Kruskal, Littlewood and others, that for one degree of freedom systems, the variation of the action is actually  $O(\epsilon^n)$  for any integer  $n$ . In fact, it is believed to be exponentially small. See §51 of Landau and Lifschitz [1976] and references in Holmes, Marsden, and Scheurle [1988].



with  $B \times T^n$ ,  $\theta$  is the canonical one-form. Then  $I^i = \int_{c_i} \theta$  define the *standard action variables* (Arnold and Avez [1968], Arnold [1978], Abraham and Marsden [1978], Chapter 5). They are smooth in  $(p, m)$ . The angle variables are constructed by integrating the flows of the actions  $I^i$ . Note that the standard actions  $I^i$  equal  $i_{\partial/\partial\theta^i} \langle \theta \rangle$ , where  $\partial/\partial\theta^i$  is the infinitesimal generator of the  $i$ -th standard basis vector in  $\mathbb{R}^n$ , thought of as the Lie algebra of  $T^n$ . Combining Theorem 10.4vi and Proposition 10.5 gives a simple proof of the following result announced in Montgomery [1988] and proved by topological arguments in Weinstein [1989b].

**10.7 Proposition** *For standard actions  $I$ , the adiabatic condition  $\langle d_M I \rangle = 0$  holds. Any two actions  $I$  and  $I'$  satisfying this adiabatic condition are related by  $I' = AI + v$ , where  $A \in \mathbf{SL}(n, \mathbb{Z})$  and  $v \in \mathbb{R}^n$  are constant.*

The second part of 10.7 is a consequence of the Arnold-Liouville theorem.

Many integrable systems are not integrable by virtue of a global torus action *i.e.*, they do not admit global action variables. For example, the spherical pendulum does not—see Duistermaat [1980]. We cannot apply 10.6 to compute, or even guarantee the existence, of the HB connection for families of such integrable systems.

This situation is salvageable, however, since every integrable system admits a local torus action, provided we delete separatrices. This means that the phase space, after deletion, is covered by an atlas of charts  $\psi_\alpha : W_\alpha \rightarrow U_\alpha \times T^n$ , where  $U_\alpha$  is an open subset of  $\mathbb{R}^n$ ,  $T^n$  is the  $n$ -torus, and the overlap maps have the form  $(\psi_\alpha \circ \psi_\beta^{-1})(x, \theta) = (f(x), \varphi_{\alpha\beta}(x) \cdot \theta)$ , where  $\varphi_{\alpha\beta}(x)$  is an affine automorphism of the torus. (An affine automorphism is one induced by an affine transformation of  $\mathbb{R}^n$ :  $\varphi_{\alpha\beta}(x) \cdot \theta = \Gamma_{\alpha\beta}(x) \cdot \theta + \lambda$ , where  $\Gamma_{\alpha\beta}(x) \in \mathbf{SL}(n, \mathbb{Z})$  and  $\theta, \lambda \in T^n = \mathbb{R}^n / \mathbb{Z}^n$ ). Since affine automorphisms are measure preserving, averaging over the torus is well-defined.

Families of integrable systems also admit local torus actions provided we delete, in addition to the separatrices, certain "bad" parameter values. These are values at which there do not exist any integrals in involution which are continuous in the parameters. (The "bad" parameter values for a family of linear oscillators are the ones for which there is a 1:1 resonance.) The charts  $\varphi_\alpha$  on the remaining open dense set are parameter dependent action-angle variables. "Averaging" is implemented by averaging over the angles in such a chart, and is globally well-defined. Since the HB connection is constructed purely through averaging, it also is well-defined. It satisfies all the properties of Proposition 10.6, provided one makes obvious notational changes (*e.g.*, the actions  $I$  are now local). For details see Montgomery [1988].

## Remarks

Perhaps the most interesting feature of these connections is their holonomy around the set  $\Sigma$  of 'bad' parameter values. The set  $\Sigma$  is in a fairly strict sense, a monopole source for the connection.

In the simplest case, alluded to above, the parameter space  $S$  is the space of real symmetric matrices. In this case,  $\Sigma$  is the set of matrices with double eigenvalues, which has codimension 2. Taking  $M$  to be a circle surrounding  $\Sigma$ , the holonomy associated with going once around  $M$  is  $-1$ . This can be viewed as either a classical or a quantum effect—the quantum effect is called the *Jahn-Teller effect*. The relevant bundle is the frame bundle associated to the canonical real line bundle over  $M = \mathbb{RP}^1$ . This frame bundle is the boundary of the Möbius strip.

We get interesting and important examples by replacing  $S$  with the set  $S_{\mathbb{C}}$  of Hermitian matrices, or with the set  $S_{\mathbb{H}}$  of quaternionic symmetric matrices. The case of  $S_{\mathbb{C}}$  is in Berry's original paper (Berry [1984]). Here  $\Sigma$  has codimension 3, and represents a Dirac monopole. The case  $S_{\mathbb{H}}$  was recently discussed by Avron, Sadun, Segert, and Simon [1989]. Here,  $\Sigma$  has codimension 5 and represents an instanton.

In the quantum case,  $\Sigma$  always represents a set of parameter values where eigenvalues collide. Kiritsis [1987] performed a general analysis of the bundles resulting from  $\Sigma$ 's of various codimensions,  $d$ . His methods are those of Steenrod obstruction theory and cohomology. His base space  $M$  is always a small sphere  $S^{d-1}$  surrounding  $\Sigma$ .

## §11 Case II: Cartan Connections and Moving Systems

The set-up of the previous section is not adequate to treat moving systems in the following sense. Consider, for instance the ball in the hoop. If one wants to apply the previous theory, one can do so using a limiting process, starting from the premise that there is a two dimensional confining potential field, then moving this potential field, computing a phase, and then taking the limit as the potential becomes an infinitely sharp, confining the particle to the one dimensional hoop. However, this method does not obviously give a technique that enables one to handle the one dimensional system *directly*. Indeed, if one attempts to do this directly as a one dimensional integrable system according to the integrable prescription, then the phase is zero. (See §12C). In this section we introduce another class of connections based on *Cartan's theory of classical spacetimes* that enables one to handle examples like the ball in the hoop in a direct manner.

### §11A Cartan Connections

Let  $(S, g)$  be a Riemannian manifold and  $M$  a space of embeddings of a given manifold  $Q$  into  $S$ . We think of  $Q$  as a given body and of  $S$  as space. Below we define a connection on the trivial bundle  $\pi : Q \times M \rightarrow M$ . The vertical subbundle of  $T(Q \times M)$  is the pull-back of  $TQ$  over  $Q \times M$ , i.e., the vector bundle  $TQ \times M \rightarrow Q \times M : (v_q, m) \mapsto (q, m)$  for  $q \in Q, m \in M$ , and  $v_q \in T_q Q$ . A tangent vector to  $M$  at  $m$  is a vector field over  $m$ , i.e., a map  $u_m : Q \rightarrow TS$  such that  $u_m(q) = T_{m(q)}S$ . Relative to the metric  $g$  on  $S$ , orthogonally project  $u_m(q)$  to  $T_{m(q)}m(Q) \in (T_q m)(T_q Q)$  and denote this vector  $u_m^T(q)$ . In this way, we have defined another element of  $T_m M$ . Pull back  $u_m^T(q)$  by  $Tm^{-1}$  to  $T_q Q$ :

$$\mathcal{U}(q) = Tm^{-1}(u_m^T(q))$$

which defines a vector field  $\mathcal{U} \in \mathfrak{X}(Q)$ . For  $Z \in \mathfrak{X}(M)$ , define  $Z^m(q) = Tm^{-1}(Z(m)^T(q))$ . The association  $m \mapsto Z(m)^T$  defines a vector field on  $M$  denoted by  $Z^T$  and so for each  $m \in M$ ,

$$Z^m = Tm^{-1} \circ Z^T(m) \in \mathfrak{X}(Q). \quad (1)$$

For moving systems, we usually take the embeddings to be restrictions of isometries of  $S$  to  $S$ . However, in the general theory, this need not be the case. For instance, consider the embeddings corresponding to blowing up a sphere by rescaling.

**11.1 Definition** *The Cartan connection on  $\pi : Q \times M \rightarrow M$  is given by the one-form  $\gamma_c \in \Omega^1(Q \times M; \ker T\pi)$  defined by*

$$\gamma_c(q, m)(v_q, u_m) = (v_q + (Tm^{-1} \circ u_m^T)(q), 0) , \quad (2)$$

where  $u_m^T$  is the pointwise  $g$ -orthogonal projection of  $u_m \in T_m M$  on  $T(m(Q))$ .

The horizontal subspace at  $(q, m)$  of this connection is given by

$$H_{(q,m)} = \ker \gamma_c(q, m) = \{(-(Tm^{-1} \circ u_m^T)(q), u_m) \mid u_m \in T_m M\}$$

and so  $T(Q \times M) = H \oplus V$ . Thus the Cartan connection defines an Ehresmann connection on  $\pi : Q \times M \rightarrow M$ . By (1) and the expression of the horizontal subspace, we see that the horizontal lift of a vector field  $Z \in \mathfrak{X}(M)$  is given by

$$(\text{hor}_c Z)(q, m) = (-(Tm^{-1} \circ Z^T(m))(q), Z_m) . \quad (3)$$

The Cartan connection induces a connection  $\gamma_0$  on  $\rho : T^*Q \times M \rightarrow M$  as follows: If  $\rho : T^*Q \times M \rightarrow M$  is the trivial bundle, then the fiber of the vertical subbundle  $\ker T\rho$  at  $(\alpha_q, m)$  is  $T_{\alpha_q}(T^*Q) \times 0_m$ . Define the *induced Cartan connection*  $\gamma_0 \in \Omega^1(T^*Q \times M ; \ker T\rho)$  by

$$\gamma_0(\alpha_q, m)(U_{\alpha_q}, u_m) = (U_{\alpha_q} + X_{\mathcal{P}u_m}(\alpha_q), 0_m) \quad (4)$$

where  $\mathcal{P}u_m$  is the momentum function of  $Tm^{-1} \circ u_m^T \in \mathfrak{X}(Q)$ , i.e.,

$$(\mathcal{P}u_m)(\alpha_q) = \alpha_q \cdot (Tm^{-1} \circ u_m^T)(q) . \quad (5)$$

The horizontal lift of  $Z \in \mathfrak{X}(M)$  relative to the Cartan connection is thus

$$(\text{hor}_0 Z)(\alpha_q, m) = (-X_{\mathcal{P}Z(m)}(\alpha_q), Z(m)) . \quad (6)$$

**11.2 Proposition** *The induced Cartan connection on  $\rho : T^*Q \times M \rightarrow M$  is a Poisson-Ehresmann connection.*

**Proof** This is immediate, since the first component of  $\text{hor}_0 Z$  is given by a Hamiltonian vector field and the Poisson bracket on  $T^*Q \times M$  is taken fiberwise. ■

### §11B The Cartan-Hannay-Berry Connection

**11.3 Theorem** Assume that a Lie group  $G$  acts on  $T^*Q$  on the left with equivariant momentum map  $I: T^*Q \rightarrow \mathfrak{g}^*$ . Then  $G$  defines a family of Hamiltonian actions on  $T^*Q \times M$  by letting  $G$  act trivially on  $M$ . Its parametrized momentum map is simply  $I$  thought of as a function of two variables, independent of  $M$ .

- i The adiabatic condition  $\langle D_0 I \rangle = 0$  holds.
- ii The induced Cartan connection on  $T^*Q \times M$  defines the Hannay-Berry connection given by the connection one-form

$$\gamma(\alpha_q, m) \cdot (U_{\alpha_q}, u_m) = (U_{\alpha_q} + X_{\langle \mathcal{P} u_m \rangle}(\alpha_q), 0).$$

Its horizontal lift for  $Z \in \mathfrak{X}(M)$  has the expression

$$(\text{hor } Z)(\alpha_q, m) = (-X_{\langle \mathcal{P} Z(m) \rangle}, Z(m)),$$

and the Hamiltonian one-form is given by

$$(K \cdot Z)(\alpha_q, m) = \mathcal{P}Z(m)(\alpha_q) - \langle \mathcal{P}Z(m) \rangle(\alpha_q).$$

Parallel transport of the Hannay-Berry connection preserves the level sets of  $I$ .

**Proof** i For  $\xi \in \mathfrak{g}$ ,  $\alpha_q \in T_q^*Q$ ,  $m \in M$ ,  $U_{\alpha_q} \in T_{\alpha_q}(T^*Q)$  and  $u_m \in T_m M$ , we have

$$\begin{aligned} (D_0 I^\xi)(\alpha_q, m) \cdot (U_{\alpha_q}, u_m) &= dI^\xi(\alpha_q, m) \cdot P_{\text{hor}_0}(U_{\alpha_q}, u_m) \\ &= dI^\xi(\alpha_q, m) \cdot (-X_{\mathcal{P} u_m}(\alpha_q), u_m) \\ &= -\{I^\xi, \langle \mathcal{P} u_m \rangle\}(\alpha_q) \end{aligned}$$

so that, since the  $G$ -action is symplectic on  $T^*Q$ , we get

$$\begin{aligned} \Phi_g^*(D_0 I^\xi)(\alpha_q, m) \cdot (U_{\alpha_q}, u_m) &= (D_0 I^\xi)(g \cdot \alpha_q, m) \cdot (g \cdot U_{\alpha_q}, u_m) \\ &= -\{I^\xi, \langle \mathcal{P} u_m \rangle\}(g \cdot \alpha_q) \\ &= -g^* \{I^\xi, \mathcal{P} u_m\}(\alpha_q), \end{aligned}$$

where  $\Phi_g$  denotes the  $G$ -action on  $T^*Q \times M$  and  $g \cdot \alpha_q, g \cdot U_{\alpha_q}$ , the corresponding actions on  $T^*Q$  and  $T(T^*Q)$ . Thus,

$$\begin{aligned}
\langle D_0 I^\xi \rangle(\alpha_q, m) \cdot \langle U_{\alpha_q}, u_m \rangle &= \frac{1}{|G|} \int_G \Phi_g^* (D_0 I^\xi)(\alpha_q, m) \cdot \langle U_{\alpha_q}, u_m \rangle dg \\
&= - \frac{1}{|G|} \int_G \Phi_g^* \{I^\xi, \mathcal{P}u_m\}(\alpha_q) dg \\
&= - \langle \{I^\xi, \mathcal{P}u_m\} \rangle(\alpha_q) .
\end{aligned}$$

Part i of the proposition now follows from the following.

**11.4 Lemma** *Let  $\pi : E \rightarrow M$  be a Poisson fiber bundle endowed with a family of Hamiltonian  $G$ -actions with equivariant parametrized momentum map  $I : E \rightarrow \mathfrak{g}^*$ . Then for any  $f : E \rightarrow \mathbb{R}$  we have  $\langle \{I^\xi, f\} \rangle = 0$  for all  $\xi \in \mathfrak{g}$ .*

**Proof of lemma** For  $g \in G$ , denote by  $g \cdot \xi = \text{Ad}_g \xi$  the adjoint action of  $G$  on  $\mathfrak{g}$ . We prove the lemma first if  $G$  is abelian. Then  $g \cdot \xi = \xi$  for all  $g \in G$ ,  $\xi \in \mathfrak{g}$  so that

$$\begin{aligned}
\langle \{I^\xi, f\} \rangle &= \frac{1}{|G|} \int_G \Phi_g^* \{I^\xi, f\} dg = \frac{1}{|G|} \int_G \{ \Phi_g^* I^\xi, \Phi_g^* f \} dg \\
&= \frac{1}{|G|} \int_G \{ \Phi_g^* \xi, \Phi_g^* f \} dg = \frac{1}{|G|} \int_G \{ I^\xi, \Phi_g^* f \} dg = \{ I^\xi, \langle f \rangle \} = 0
\end{aligned}$$

by  $G$ -invariance of  $\langle f \rangle$  and conservation of momentum maps.

Next we prove the general case. Fix  $\xi \in \mathfrak{g}$  and let  $T$  denote the maximal torus containing  $\exp \xi$ ; this is always possible since  $G$  is assumed to be compact. By Fubini's theorem we get for any  $\varphi : E \rightarrow \mathbb{R}$

$$\begin{aligned}
\langle \varphi \rangle(p) &= \frac{1}{|G|} \int_G \varphi(g \cdot p) dg = \frac{1}{|G|} \int_{Tg \in G/T} \left( \int_{t \in T} (h \cdot p) dh \right) d(Tg) \\
&= \frac{1}{|G|} \int_{Tg \in G/T} \left( \int_{t \in T} (f \cdot g \cdot p) dp \right) d(Tg)
\end{aligned}$$

where we denote by  $dt, dh, d(Tg)$  the induced Haar measures on  $T, Tg$  and  $G/T$  respectively. Now apply this formula to  $\varphi = \{I^\xi, f\}$ . Then

$$\varphi(t \cdot g \cdot p) = (\Phi_g^* \Phi_t^* \{I^\xi, f\})(p)$$

and hence

$$\langle \{I^\xi, f\} \rangle(p) = \frac{1}{|G|} \int_G \Phi_g^* \{I^\xi, f\}(p) dg = \frac{1}{|G|} \int_{Tg \in G/T} \left( \int_{t \in T} \Phi_t^* \{I^\xi, f\}(p) dt \right) d(Tg)$$

The interior fiber integral is zero by the first step so that the entire integral vanishes. ▼

ii For  $Z \in \mathfrak{X}(M)$ ,

$$(\text{hor } Z)(\alpha_q, m) = \langle (\text{hor}_0 Z)(\alpha_q, m) \rangle = \frac{1}{|G|} \int_G \Phi_g^* \left( -X_{\mathcal{P}Z(m)}(\alpha_q), Z(m) \right) dg$$

$$= \langle -X_{\langle \mathcal{P}Z(m) \rangle}(\alpha_q), Z(m) \rangle$$

and hence

$$(\text{hor } Z - \text{hor}_0 Z)(\alpha_q, m) = (X_{\mathcal{P}Z(m)} - \langle \mathcal{P}Z(m) \rangle, 0)$$

In view of i, the Hannay-Berry connection has parallel transport preserving the level sets of  $\mathbf{I}$ . ■

## §12 The Cartan Angles; the Ball in the Hoop and the Foucault Pendulum

In this section we compute the holonomy of the Hannay-Berry connection induced by a Cartan connection. We shall treat two examples in detail: the ball in the hoop and the Foucault pendulum. For the ball in the hoop example we will show that the Hannay angle formula  $\int_C \langle d_M \theta \rangle$  does *not* give the correct phase shift if we take the parameter space to be the space of frequencies.

### §12A Cartan Angles

Recall that the holonomy of a closed loop relative to an Ehresmann connection is the diffeomorphism of the fiber given by parallel translation. In the case of the Hannay-Berry connection induced by a Cartan connection, the fiber is  $T^*Q$ . Thus if  $c(t)$  is a closed loop of embeddings of  $Q$  in  $S$  the differential equations for the horizontal lift of  $c(t)$  in  $T^*Q$  are Hamilton's equations for the Hamiltonian  $\langle \mathcal{P}(Tc(t)^{-1} \circ c'(t)^T) \rangle$ ; see Theorem 11.3.

Marsden [1981] proved that there is a neighborhood  $V$  of  $G_\mu$  in  $T^*G_\mu \times (T^*Q)_\mu \times O_\mu$  and a neighborhood  $U$  of  $G_\mu \cdot \alpha_q$  in  $T^*Q$  and a symplectic diffeomorphism  $F: V \rightarrow U$ . Here  $G_\mu$  is the coadjoint isotropy subgroup at  $\mu = I(\alpha_q)$ ,  $O_\mu$  is the coadjoint orbit through  $\mu$  in  $\mathfrak{g}^*$ , and  $G_\mu \cdot \alpha_q$  is the  $G_\mu$ -orbit of  $\alpha_q$  in  $T^*Q$ . The composition of the momentum map with this diffeomorphism is a  $\mathfrak{g}^*$ -valued function whose level set at  $\mu$  is diffeomorphic to  $G_\mu \times (T^*Q)_\mu$ . Thus the holonomy will have  $G_\mu$ -components which we call the *Cartan angles*.

These considerations become very explicit in the case of a completely integrable system, for then  $O_\mu = \{\mu\}$  and  $(T^*Q)_\mu$  is a point. In this case  $I: T^*Q \rightarrow \mathbb{R}^n$  is the momentum map of a torus action for a completely integrable system. Assume, as in §10, that  $I$  represent global action variables with  $(\theta^1, \dots, \theta^n)$  the conjugate angle variables. If this is not the case, the theory must be reformulated in terms of local group actions as outlined at the end of §10 and treated in detail by Montgomery [1988]. By linearity of the momentum function, we have for a given embedding  $m$ ,

$$\frac{\partial}{\partial I_i} \langle \mathcal{P}(Tm^{-1} \circ u_m^T) \rangle = \langle (Tm^{-1} \circ u_m^T)^i \rangle, \quad (1)$$

where  $Tm^{-1} \circ u_m^T = (Tm^{-1} \circ u_m^T)^j \partial/\partial \theta^j$  (sum over  $j$ ).



Now let  $c(t)$ ,  $t \in [0, 1]$  be a closed loop based at  $m_0 \in M$ . By the formula for the horizontal lift of the induced Hannay-Berry connection given in Proposition 10.3ii and i, it follows that  $(\theta(t), c(t)) \in I^{-1}(\mu)$  is the horizontal lift of the loop  $c(t)$  if and only if

$$\frac{d\theta^i}{dt} = \langle Tc(t)^{-1} \circ c'(t)^T \rangle,$$

whence the holonomy of this loop is

$$\Delta\theta = \theta(1) - \theta(0) = \int_0^1 \langle Tc(t)^{-1} \circ c'(t)^T \rangle dt. \quad (2)$$

This formula is analogous in spirit to the expression of the Hannay angles discussed in §10 for integrable systems. This is why we shall call the holonomy in (2) the *Cartan angles*.

### §12B The Ball in the Hoop

Let us apply formula (2) to the ball in the hoop discussed in the introduction. Consider a not necessarily circular hoop of length  $L$  in the plane, enclosing an area  $A$ . On the hoop we consider a bead sliding without friction. As in Figure 1B-1, let  $s$  denote the arc length along the hoop measured from a reference point on the hoop and let the shape of the hoop be described by a function  $q(s)$  in a given inertial frame in space. No external forces are acting on the bead, so that its total energy equals its kinetic energy  $\frac{1}{2} m \|q'(s(t)) \dot{s}(t)\|^2 = \frac{1}{2} m \dot{s}(t)^2$  since  $\|q'(s(t))\|^2 = 1$ . Thus the Hamiltonian of this system  $H: T^*S^1 \rightarrow \mathbb{R}$  is given by

$$H(s, p_s) = \frac{1}{2m} p_s^2;$$

the solution of the corresponding Hamiltonian system is  $s(t) = \frac{1}{2m} \mu t + s_0$ , where the conserved momentum is  $p_s(t) = \mu \in \mathbb{R}$ . The system is integrable and we choose global action angle coordinates given by  $(\theta, I) = (s, p_s)$ ; we consider the  $S^1$ -action on the hoop being given by the flow of this system. (Strictly speaking, we should take  $\theta = \frac{2\pi}{L} s$  and  $I = \frac{L}{2\pi} p_s$ , so that  $\theta$  ranges from 0 to  $2\pi$ , but we shall take  $\theta$  ranging from 0 to  $L$  to simplify notation.)

Assume the hoop rotates in the plane with prescribed angular velocity  $\omega(t)$ . Consider the space  $M$  of embeddings of the given hoop into  $S = \mathbb{R}^2$  given by the possible rotated configurations of the hoop relative to a given point, say, the origin. Thus the convective velocity is  $Tc(t)^{-1} \circ c'(t) = \omega(t) \mathbf{k} \times \mathbf{q} = \omega(-y, x)$ . Therefore  $Tc(t)^{-1} \circ c'(t)^T$  represents its orthogonal

projection at  $(x, y)$  onto the tangent space of the hoop, *i.e.*,  $Tc(t)^{-1} \circ c'(t)^T = \omega(-y, x) \cdot \mathbf{q}'(s) = \omega(-y, x)(x', y') = \omega(xy' - yx')$ . Therefore, its average over the dynamics is

$$\langle Tc(t)^{-1} \circ c'(t)^T \rangle = \frac{1}{L} \int_0^L \omega(t)(x(s)y'(s) - y(s)x'(s))ds = \frac{\omega(t)}{L} \int_{\text{hoop}} (xdy - ydx) = \omega(t) \frac{2A}{L}$$

by Green's theorem. Therefore the Cartan angle equals

$$\Delta\theta = \int_0^1 \langle Tc(t)^{-1} \circ c'(t)^T \rangle dt = \frac{2A}{L} \int_0^1 \omega(t) dt = \frac{2A}{L} \int_0^1 d\alpha(t) dt = \frac{2A}{L} (\alpha(1) - \alpha(0))$$

where  $\alpha(t)$  is the angular variable whose time derivative is  $\omega(t)$ . Since  $\alpha(1) - \alpha(0)$  is the length of the circle of radius 1,  $\alpha(1) - \alpha(0) = 2\pi$  and so

$$\Delta\theta = \frac{4\pi A}{L} \quad (3)$$

the phase shift found by the elementary methods given in the introduction.

Treating the ball in the hoop example as a family of completely integrable systems depending on a parameter and computing the associated Hannay angle yields zero. Thus, the geometric phase relative to the Hannay-Berry connection induced by the trivial connection from §9 is zero and the entire phase is a purely dynamic phase obtained by reconstruction, as in §2 and §3. As in §1, the Lagrangian of the rotating system with angular velocity  $\omega$  is given by

$$L(s, \dot{s}; \omega) = \frac{1}{2} m \|\mathbf{q}'(s)\dot{s} + \omega \times \mathbf{q}(s)\|^2. \quad (4)$$

The Legendre transformation  $p_s = \frac{\partial L}{\partial \dot{s}} = m[\dot{s} + (\omega \times \mathbf{q}(s)) \cdot \mathbf{q}'(s)]$  gives the Hamiltonian

$$H(s, p_s; \omega) = \frac{1}{2m} [p_s - m(\omega \times \mathbf{q}(s)) \cdot \mathbf{q}'(s)]^2 - \frac{1}{2} m \|\omega \times \mathbf{q}(s)\|^2. \quad (5)$$

For each fixed  $\omega$ , the Hamiltonian system on  $T^*S^1$  given by  $H$  is completely integrable since it is one degree of freedom. Thus  $H$  defines a completely integrable system depending on the frequency parameter  $\omega$ . The space of  $\omega$ 's form a real line  $\mathbb{R}$ . The holonomy of any bundle over  $\mathbb{R}$  is necessarily trivial. (This is because any loop in  $\mathbb{R}$  comes back on itself.) Hence the Hannay angle is zero.

### §12C The Foucault Pendulum

As in §10, we cannot literally apply the formula for the Cartan angles since, in this example, it is known that global action-angle variables do not exist (Duistermaat [1980]). This is not really a serious problem, since we can restrict to the part of the phase space corresponding to stable oscillations. The Foucault pendulum is a spherical pendulum at co-latitude  $\alpha$  on the surface of the Earth. Denote by  $\mathbf{q}$  the position of the pendulum on the sphere of radius  $\ell = \|\mathbf{q}\|$ , the length of the pendulum arm (see Figure 12C-1).

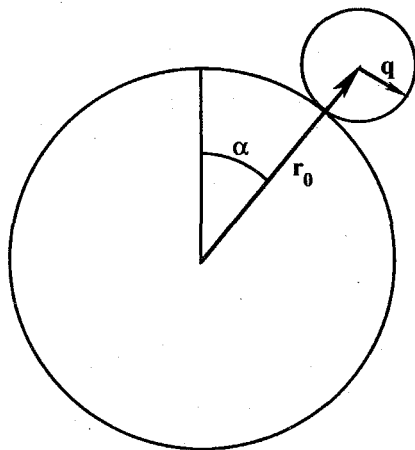


Figure 12C-1

Let  $\mathbf{r}_0$  denote the vector from the center of the Earth to the point of suspension of the pendulum. The position of the tip of the pendulum in space is  $R_t(\mathbf{r}_0 + \mathbf{q})$ , where  $R_t$  is the rotation about the  $Oz$ -axis. Let  $\omega = T/2\pi$  denote the angular velocity of the Earth's rotation. The potential energy of the pendulum is  $V(\mathbf{q}) = mg\ell\mathbf{q} \cdot \hat{\mathbf{r}}_0$ , where  $\hat{\mathbf{r}}_0 = \mathbf{r}_0/\|\mathbf{r}_0\|$ . The velocity of the pendulum's tip in space is

$$R_t \dot{\mathbf{q}} + R_t[\omega \times (\hat{\mathbf{r}}_0 + \mathbf{q})]$$

where we identify  $\omega$  with the vector  $\omega \mathbf{k}$ . The Lagrangian is therefore given by

$$L = \frac{1}{2} m \|\omega \times (\hat{\mathbf{r}}_0 + \mathbf{q}) + \dot{\mathbf{q}}\|^2 - V(\mathbf{q})$$

so the Legendre transformation gives

$$\mathbf{p} = m(\dot{\mathbf{q}} + \omega \times (\hat{\mathbf{r}}_0 + \mathbf{q}))^T = m[\dot{\mathbf{q}} + (\omega \times (\hat{\mathbf{r}}_0 + \mathbf{q}))^T]$$

by identifying  $T^*S^2$  with  $TS^2$  via the Euclidean metric, where  $(\omega \times (\hat{r}_0 + q))^T$  is the tangential component of  $\omega \times (\hat{r}_0 + q)$ . Then

$$\dot{q} = \frac{1}{m} p - [\omega \times (\hat{r}_0 + q)]^T$$

and so

$$\dot{q} + \omega \times (\hat{r}_0 + q) = \frac{1}{m} p + [\omega \times (\hat{r}_0 + q)]^\perp,$$

where  $(\omega \times (\hat{r}_0 + q))^\perp$  denotes the normal component of  $\omega \times (\hat{r}_0 + q)$  to the sphere. Therefore the Hamiltonian is

$$\begin{aligned} H = p\dot{q} - L &= \frac{1}{m} \|p\|^2 - \frac{1}{m} p \cdot (\omega \times (\hat{r}_0 + q))^T \\ &\quad - \left( \frac{1}{2m} \|p\|^2 + \frac{m}{2} \|(\omega \times (\hat{r}_0 + q))^\perp\|^2 - V(q) \right) \\ &= \frac{1}{2m} \|p\|^2 + V(q) - \mathcal{P} - \frac{m}{2} \|(\omega \times (\hat{r}_0 + q))^\perp\|^2, \end{aligned}$$

where

$$\mathcal{P} = \frac{1}{m} p \cdot (\omega \times (r_0 + q))^T = \frac{1}{m} p \cdot (\omega \times (r_0 + q))$$

since  $p$  is tangential.

Let  $M$  be the space of embeddings of the sphere of radius  $\ell$  in  $\mathbb{R}^3$  tangent to the Earth at co-latitude  $\alpha$ , as in Figure 12C-1. We recognize in  $\mathcal{P}$  the Hamiltonian defining the induced Cartan connection on  $T^*S^2 \times S^2$  (formula (4) of §11).

**Remark** Since  $(\omega \times q)^\perp = 0$ , we have  $[\omega \times (r_0 \times q)]^\perp = (\omega \times r_0)^\perp = \text{constant}$ . In the equations of motion one can drop this constant. Had we considered a Foucault pendulum on an ellipsoid, this term would not be constant, but it would be of order  $\omega^2$ , which is the general case. As explained in the introduction, applying the averaging principle, one would ignore this term.

Let  $S^1$  act on the phase space of the pendulum by rotation about the  $r_0$  axis. We want to compute the part of the holonomy corresponding to this  $S^1$ -action. According to Proposition 11.4, the horizontal lift of the induced Hannay-Berry connection is given by the Hamiltonian vector field of  $\mathcal{P}$ , i.e., by  $\langle p, [\omega \times (r_0 \times q)] \rangle$ . If  $v$  is any constant vector,

$$\langle p, v \rangle = \langle p, \hat{r}_0 \rangle (v \cdot \hat{r}) \quad (8)$$

since the  $S^1$ -action over which we average has  $r_0$  as its axis of rotation. Setting  $v = \omega \times r_0$  this implies that

$$\langle p, (\omega \times r_0) \rangle = 0,$$

and hence

$$\langle p, [\omega \times (r_0 + q)] \rangle = \langle p, (\omega \times q) \rangle.$$

Let  $I = \mathbf{p} \cdot (\mathbf{r}_0 \times \mathbf{q})$  be the momentum map of the  $S^1$ -action (rotation about  $\mathbf{r}_0$ ). By (8)

$$\langle \mathbf{p} \cdot (\omega \times \mathbf{q}) \rangle = \langle \omega \cdot (\mathbf{q} \times \mathbf{p}) \rangle = [(\mathbf{q} \times \mathbf{p}) \cdot \mathbf{r}_0] \langle \omega \cdot \hat{\mathbf{r}}_0 \rangle = I \omega \cos \alpha$$

since  $\omega \cdot \hat{\mathbf{r}}_0$  is constant. Thus,

$$\langle \mathcal{P} \rangle = I \omega \cos \alpha$$

and so the horizontal lift of  $\omega$  is given by  $(-X_{\langle \mathcal{P} \rangle}, \omega) = \left( -\omega \cos \alpha \frac{\partial}{\partial \theta}, \omega \right)$ , where  $\frac{\partial}{\partial \theta}$  is the infinitesimal generator of the  $S^1$ -action corresponding to  $I$ . Therefore the horizontal lift of a curve in  $\theta$  is given by the differential equation

$$\dot{\theta} = -\omega \cos \alpha$$

so that if this curve is a loop parametrized on  $[0, T/2\pi]$  we get

$$\theta(T) - \theta(0) = - \int_0^T \omega \cos \alpha \, dt = -\omega T \cos \alpha = -2\pi \cos \alpha$$

which is the deviation of the plane of oscillation in the laboratory frame (*i.e.*, a frame fixed on the Earth) of the Foucault pendulum during 24 hours.

For example, if we are at the equator, where  $\alpha = \pi/2$ , there is no deviation. If we are at the North Pole, where  $\alpha = 0$ , the plane rotates in the opposite direction to that of the Earth's rotation, performing a full circular motion. This is the usual Foucault result for a lab frame attached to the Earth. The laboratory and inertial frames are related by  $\mathbf{q}_{\text{inertial}} = \mathbf{R}_t(\mathbf{r}_0 + \mathbf{q}_{\text{lab}})$ . As long as we regard angles as taking values in the circle, where  $0$  is identified with  $2\pi$ , then the answer is the same in the inertial frame as in the lab frame. (It seems that there is no way of keeping track of the full angular change in the inertial frame, since this would require a reference line in each tangent space to the Earth relative to which we measure the angle. Since there are no nonvanishing vector fields on  $S^2$ , this is not possible.)

### §13 Induced Connections

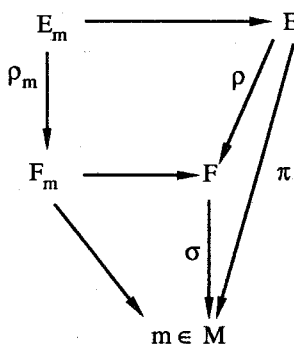
Assume that  $\pi : E \rightarrow M$ ,  $\rho : E \rightarrow F$ , and  $\sigma : F \rightarrow M$  are surjective submersions such that  $\pi = \sigma \circ \rho$  and  $\rho_m : E_m \rightarrow F_m$  defined by  $\rho_m = \rho|_{E_m}$  is a surjective submersion for all  $m \in M$ . Let  $E_m = \pi^{-1}(m)$  and  $F_m = \sigma^{-1}(m)$  denote the fibers of  $\pi$  and  $\sigma$  respectively. Assume there is a connection  $\gamma^M \in \Omega^1(E; V^M)$  with horizontal lift  $\text{hor}^M : TM \rightarrow \mathfrak{X}_{\text{hor}}(E; M)$  for  $\pi : E \rightarrow M$ , where  $V^M = \ker T\pi$  is the vertical subbundle given by  $\pi$ . Denote by  $H^M = \ker \gamma^M$  the horizontal subbundle of  $TE$  defined by  $\gamma^M$ . In addition, assume that for each  $m \in M$  the bundles  $\rho_m : E_m \rightarrow F_m$  have connections  $\gamma^m \in \Omega^1(E_m; V^m)$  with horizontal lifts  $\text{hor}^m : TF_m \rightarrow \mathfrak{X}_{\text{hor}}(E_m; V^m)$ , where  $V^m = \ker T\rho_m$  is the vertical subbundle of  $TE_m$  given by  $\rho_m$ . Let  $H^m = \ker \gamma^m$  be the horizontal subbundle of  $TE_m$  defined by  $\gamma^m$  and assume that the connections  $\gamma^m$  depend smoothly on  $m$  in the sense that if  $Z : M \times F \rightarrow TF$  is a smooth map satisfying  $Z(m, \cdot) \in \mathfrak{X}(F_m)$  for all  $m \in M$ , then the mapping

$$e \in E \mapsto (\text{hor}^m Z(m, \rho_m(e)))(e) \in TE, \text{ for } m = \pi(e),$$

is a smooth vector field on  $E$ . Equivalently, in terms of the connection forms  $\gamma^m$ , smoothness means that

$$e \in E \mapsto \gamma^m(Z(m, \rho_m(e))) \in V^m, \text{ for } m = \pi(e),$$

is a smooth vector field on  $E$ ; its value at each  $e$  is vertical relative to  $\rho_m$ . Let  $V_F = \ker T\rho$  be the vertical subbundle of  $TE$  relative to  $\rho$ . The maps involved are summarized in the following diagram



whose horizontal arrows are inclusions.

**13.1 Theorem** The map  $\gamma^F(e) = \gamma^m(e) \circ \gamma^M(e)$  defines a connection form  $\gamma^F \in \Omega^1(E; V^F)$  for the bundle  $\rho : E \rightarrow F$ . The pull-back of  $\gamma^F$  to each  $E_m$  coincides with  $\gamma^m$ . If  $H^F = \ker \gamma^F$  denotes the horizontal subbundle of  $TE$  defined by  $\gamma^F$ , then

$$H_e^F = H_e^m \oplus H_e^M, \quad V_e^F = V_e^m \subset V_e^M, \quad \text{and} \quad H_e^F \cap V_e^M = H_e^m.$$

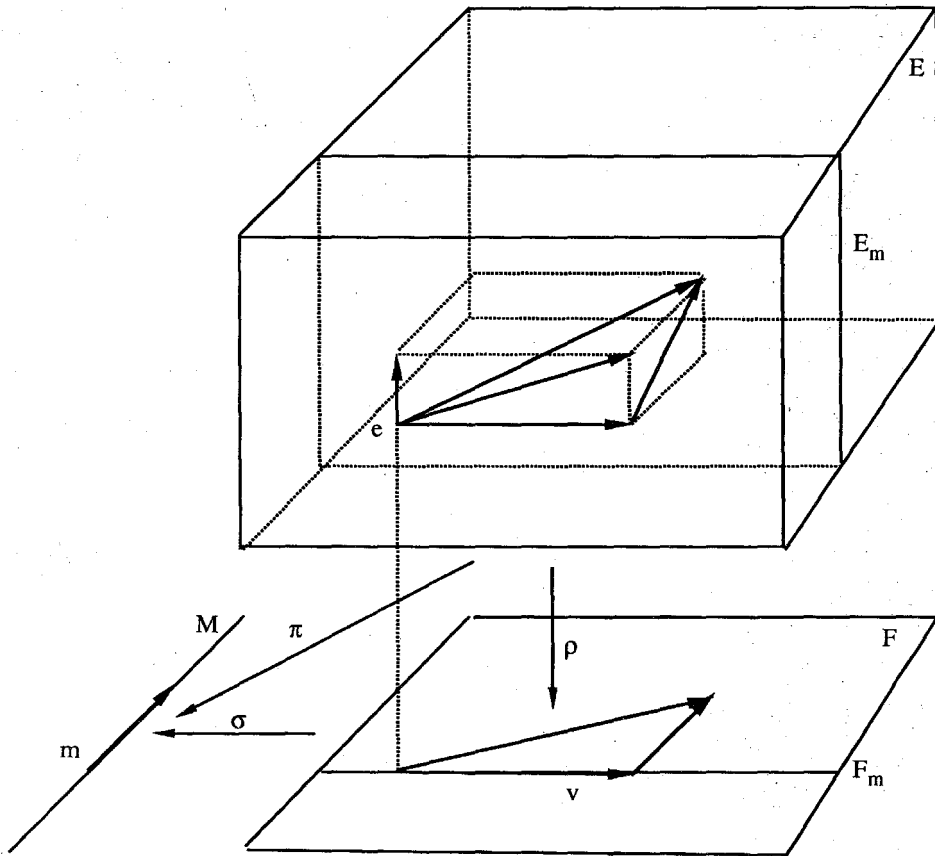


Figure 13-1

**Proof** By definition,  $\gamma^M(e)(v) \in V_e^M = \ker T_e \pi = T_e E_m$  for  $m = \pi(e)$ . Thus  $\gamma^m(e) \circ \gamma^M(e)$  defines a map  $\gamma^F(e) : T_e E \rightarrow V_e^m$ . Since  $E_m \subset E$  and  $\rho_m = \rho|_{E_m}$ , we obtain  $V_e^m = \ker T_e \rho_m \subset \ker T_e \rho = V_e^F$ . Conversely, if  $v \in V_e^F$ , then  $T_e \rho(v) = 0$ , so that  $0 = (T_{\rho(e)} \sigma \circ T_e \rho)(v) = T_e \pi(v)$ ,

*i.e.*,  $v \in \ker T_e \pi = TE_m$ . On  $E_m$ ,  $\rho$  and  $\rho_m$  coincide, so  $T_e \rho_m(v) = 0$ ; *i.e.*,  $v \in V_e^m$ . This shows that  $V_e^m = V_e^F$  and hence  $\gamma^F(e) : T_e E \rightarrow V_e^F$  is the identity on  $V_e^F$ . Since  $\sigma \circ \rho = \pi$ , we see that  $V_e^F \subset V_e^M$  and hence if  $Z$  is any smooth vector field on  $E$  defined on a neighborhood of  $e$ , then  $\gamma(Z)$  is a smooth vertical vector field (on this neighborhood) relative to  $\pi$  and  $e \mapsto \gamma^F(Z)(e) = \gamma^m(e)(\gamma(Z)(e))$  is smooth by the smoothness condition on the  $\gamma^m$ . Thus  $\gamma^F \in \Omega^1(E; V^F)$  is a connection one-form. Note that  $T_e E = H_e^M \oplus V_e^M = H_e^M \oplus T_e E_m = H_e^M \oplus H_e^m \oplus V_e^F$ , whence  $H_e^F = H_e^M \oplus H_e^m$ . Intersect this relation with  $V_e^M$  to get  $H_e^F \cap V_e^M = H_e^m$  since  $H_e^m \subset V_e^M = T_e E_m$ . Finally, if  $v \in T_e E_m = V_e^M$ , then  $\gamma^M(e)(v) = v$  and thus  $\gamma^F(e)(v) = \gamma^m(e)(v)$ , *i.e.*, the pull-back of  $\gamma^F$  to  $E_m$  equals  $\gamma^m$ . ■

We compute the curvature  $\Omega^F$  of  $\gamma^F \in \Omega^1(E; V^F)$  in terms of the curvatures  $\Omega^M$  of  $\gamma^M \in \Omega^1(E; V^M)$  and  $\Omega^m$  of  $\gamma^m \in \Omega^1(E_m; V^m)$ . If  $X \in \mathfrak{X}(E)$ , denote by  $X^F$  and  $X^M$  the horizontal projections of  $X$  given by  $\gamma^F$  and  $\gamma^M$ . By the previous theorem,  $X^F$  splits into two components: one in  $H^m$ , the other in  $H^M$ . Note that  ${}_m X := \langle \gamma^M, X \rangle|_{E_m}$  defines a vector field on  $E_m$ . Writing

$$X^F(e) = X^1(e) + X^2(e) \in H_e^m \oplus H_e^M, \quad (1)$$

we get  $T_e \pi(X^F(e)) = T_e \pi(X^2(e))$  since  $T_e \pi(X^1(e)) = 0$  *i.e.*,  $X^1(e) \in H_e^m \subset V_e^M = \ker T_e \pi$ . Since  $T_e \rho(X(e)) = T_e \rho(X^F(e))$  and  $\pi = \sigma \circ \rho$ , we conclude that

$$T_e \pi(X^F(e)) = (T_{\rho(e)} \sigma \circ T_e \rho)(X^F(e)) = (T_{\rho(e)} \sigma \circ T_e \rho)(X(e)) = T_e \pi(X(e)) = T_e \pi(X^2(e)),$$

*i.e.*,  $X(e)$  and  $X^2(e)$  have the same horizontal component in the splitting  $T_e E = H_e^M \oplus V_e^M$ . Since  $X^2(e) \in H_e^m$ , we conclude that  $X^2(e) = X^M(e)$ . Applying  $\gamma^M$  to (1) we get

$$\langle \gamma^M, X^F \rangle(e) = \langle \gamma^M, X^1 \rangle(e) \quad (2)$$

since  $\gamma^M(e)(X^M(e)) = 0$ . However,  $\langle \gamma^M, X^F \rangle = \langle \gamma^M, X \rangle$  since  $\gamma^M$  is the identity on  $V^M$  and  $V^F \subset V^M$ . On the other hand, since  $X^1(e) \in H_e^m \subset V_e^M$ , we have  $\langle \gamma^M, X^1 \rangle(e) = X^1(e)$  and so by (2),  $X^1(e) = \langle \gamma^M, X \rangle(e)$ . We conclude that

$$X^F = \langle \gamma^M, X \rangle + X^M, \quad (3)$$

or if  $m = \pi(e)$ ,

$$X^F(e) = {}_m X(e) + X^M(e). \quad (4)$$

By formula (6) of §4, the curvature of  $\gamma^F$  equals

$$\begin{aligned} \Omega^F(X, Y)(e) &= -\langle \gamma^F, [X^F, Y^F] \rangle(e) = -\langle \gamma^F, [\langle \gamma^M, X \rangle + X^M, \langle \gamma^M, Y \rangle + Y^M] \rangle(e) \\ &= -(\gamma^m(e) \circ \gamma^M(e))([ \langle \gamma^M, X \rangle, \langle \gamma^M, Y \rangle ](e)) \end{aligned}$$



$$\begin{aligned}
& + [\langle \gamma^M, X \rangle, Y^M](e) + [X^M, \langle \gamma^M, Y \rangle](e) + [X^M, Y^M](e)) \\
= & - (\gamma^m(e) \circ \gamma^M(e))([\langle \gamma^M, X \rangle, \langle \gamma^M, Y \rangle](e)) \\
& - (\gamma^m(e) \circ \gamma^M(e))([\langle \gamma^M, X \rangle, Y^M](e) + [X^M, \langle \gamma^M, Y \rangle](e)) \\
& - (\gamma^m(e) \circ \gamma^M(e))([X^M, Y^M](e)) .
\end{aligned} \tag{5}$$

where  $X, Y \in \mathfrak{X}(E)$ . Since  $\langle \gamma^M, X \rangle$  and  $\langle \gamma^M, Y \rangle$  restricted to each  $E_m$  are tangent to  $E_m$ , it follows that their bracket is again tangent to each  $E_m$ , i.e.,  $[\langle \gamma^M, X \rangle, \langle \gamma^M, Y \rangle](e) \in T_e E_m = V_e^M$ , the first term in (5) becomes

$$\Omega^m(\langle \gamma^M, X \rangle, \langle \gamma^M, Y \rangle)(e), \text{ for } m = \pi(e). \tag{6}$$

The last term in (5) is

$$\gamma^m(e)(\Omega^M(X, Y)(e)). \tag{7}$$

The second term in (5) can be simplified if there is an additional compatibility condition between  $\gamma^M$  and the family  $\{\gamma^m\}$ . Note that  $\langle \gamma^M, X \rangle$  is a vector field on  $E$  whose value at  $e$  is in  $H_e^M$ . There are two obvious bracket conditions one can request: for any vector field  $X$  on  $E$  whose value at  $e$  is in  $H_e^m$  and any  $Y \in \mathfrak{X}_{\text{hor}}(E; M)$ , the bracket  $[X, Y]$  is

- i either in  $\mathfrak{X}_{\text{hor}}(E; M)$ , or
- ii its value at  $e$  is in  $H_e^m$ .

In case i the second term vanishes since  $\gamma^M(e)$  has kernel  $H_e^M$ . In case ii, since  $H_e^m \subset V_e^M$ ,  $\gamma^M(e)$  is the identity on  $H_e^m$ , and then  $\gamma^m(e)$  kills the second term, so in both cases the second term in (5) vanishes. We have proved the following:

**13.2 Proposition** Assume either condition i or ii holds. Then the curvature  $\Omega^F \in \Omega^2(E; V^F)$  is given by

$$\Omega^F(X, Y)(e) = \Omega^m(\langle \gamma^M, X \rangle, \langle \gamma^M, Y \rangle)(e) + \gamma^m(e)(\Omega^M(X, Y)(e)) \tag{8}$$

for any  $X, Y \in \mathfrak{X}(E)$ .

To compute the induced map  $\bar{\Omega}^F: \mathfrak{X}(F) \times \mathfrak{X}(F) \rightarrow \mathfrak{X}_{\text{vert}}(E; F)$  on the base, we use (8) and the horizontal lift. Let  $\text{hor}^M: \mathfrak{X}(M) \rightarrow \mathfrak{X}_{\text{hor}}(E; M)$ ,  $\text{hor}^m: \mathfrak{X}(F_m) \rightarrow \mathfrak{X}_{\text{hor}}(E_m; F_m)$ , and  $\text{hor}^F: \mathfrak{X}(F) \rightarrow \mathfrak{X}_{\text{hor}}(E; F)$  be the horizontal lifts given by  $\gamma^M$ ,  $\gamma^m$ , and  $\gamma^F$  respectively. If  $U \in \mathfrak{X}(F)$ , then  $\text{hor}^F U \in \mathfrak{X}_{\text{hor}}(E; F)$  so that by (3)

$$\text{hor}^F U = \langle \gamma^M, \text{hor}^F U \rangle + (\text{hor}^F U)^M,$$

with  $(\text{hor}^F U)^M(e) \in H_e^M$  and  $\langle \gamma^M, \text{hor}^F U \rangle(e) \in H_e^{\pi(e)} \subset V_e^M$ . Thus, if  $W \in \mathfrak{X}(F)$ , we get

$$\begin{aligned}\bar{\Omega}^F(U, W)(e) &= \Omega^F(\text{hor}^F U, \text{hor}^F W)(e) \\ &= \Omega^m(e)(\langle \gamma^M, \text{hor}^F U \rangle, \langle \gamma^M, \text{hor}^F W \rangle)(e) + \gamma^m(e)(\Omega^M((\text{hor}^F U)^M, (\text{hor}^F W)^M)(e)).\end{aligned}\quad (9)$$

This expression can be simplified in the following way. We show that

$$(\text{hor}^F U)^M(e) = \text{hor}_e^M(T_{\rho(e)}\sigma \cdot U(e)). \quad (10)$$

Since both sides are in  $H_e^M$ , it suffices to show that  $T_e\pi((\text{hor}^F U)^M(e)) = T_{\rho(e)}\sigma \cdot U(e)$ , which is proved as follows. Since  $\pi = \sigma \circ \rho$  and  $T_e\rho((\text{hor}^F U)(e)) = U(e)$ , we get  $T_e\pi((\text{hor}^F U)^M(e)) = (T_{\rho(e)}\sigma \circ T_e\rho)((\text{hor}^F U)(e)) = T_{\rho(e)}\sigma \cdot U(e)$ . By (10), the last term in (9) equals

$$\gamma^m(e)(\bar{\Omega}^M(e)(T_{\rho(e)}\sigma \cdot U(e), T_{\rho(e)}\sigma \cdot W(e))). \quad (11)$$

For the first term in (9), write  $\langle \gamma^M, \text{hor}^F U \rangle(e)$  as  $\text{hor}_e^m$  of the vector  $T_e\rho(\langle \gamma^M, \text{hor}^F U \rangle(e))$  in  $F_m = \sigma^{-1}(m)$ . But

$$T_e\rho(T_e E) = T_e\rho(V_e^F \oplus H_e^F) = T_e\rho(H_e^F) = T_e\rho(H_e^m \oplus H_e^M) = T_e\rho(H_e^m) \oplus T_e\rho(H_e^M) = T_{\rho(e)}F.$$

Note that  $T_e\rho(H_e^m) = \ker T_{\rho(e)}\sigma$ . Indeed  $(T_{\rho(e)}\sigma \circ T_e\rho)(H_e^m) = T_e\pi(H_e^m) = 0$  since  $V_e^M \supset H_e^m$ , so that  $T_e\rho(H_e^m) \subset \ker T_{\rho(e)}\sigma$ . Also  $\ker T_{\rho(e)}\sigma \cap T_e\rho(H_e^M) = \{0\}$ , for if  $v \in H_e^M$  and  $(T_{\rho(e)}\sigma \circ T_e\rho)(v) = 0$ , then  $T_e\pi(v) = 0$ , whence  $v \in V_e^M \cap H_e^M = \{0\}$ .

Since  $T\rho(H^M)$  is a smooth subbundle of  $TF$ , being the image of a subbundle by a vector bundle map, it follows that

$$TF = \ker T\sigma \oplus T\rho(H^M), \quad (12)$$

i.e., there is an induced connection  $\gamma^{M,F} \in \Omega^1(F; \ker T\sigma)$ . In addition, since  $T_e\rho(H_e^m) = \ker T_e\sigma$ , we get

$$T_e\rho(\langle \gamma^M, \text{hor}^F U \rangle(e)) = \gamma^{M,F}(e)(U(e)). \quad (13)$$

Thus the first term in (9) equals

$$\bar{\Omega}^m(e)(\gamma^{M,F}(e) \cdot U(e), \gamma^{M,F}(e) \cdot W(e)) \quad (14)$$

and so by (11) and (13),

$$\begin{aligned}\bar{\Omega}^F(U, W)(e) &= \bar{\Omega}^m(e)(\gamma^{M,F}(e) \cdot U(e), \gamma^{M,F}(e) \cdot W(e)) \\ &\quad + \gamma^m(e)(\bar{\Omega}^M(e)(T_{\rho(e)}\sigma \cdot U(e), T_{\rho(e)}\sigma \cdot W(e))).\end{aligned}\quad (15)$$

We have proved the following.

**13.3 Theorem** Assuming the hypotheses of Theorem 13.1, there is an induced connection  $\gamma^{M,F} \in \Omega^1(F; \ker T\sigma)$  on  $F$  whose horizontal subbundle is given by  $T\rho(H^M)$ . Under the hypotheses of 13.2, the curvature of the connection  $\gamma^F$  induces a map  $\bar{\Omega}^F$  on the base  $F$  of the surjective submersion  $\rho: E \rightarrow F$  given by (15) for any  $U, V \in \mathfrak{X}(F)$ .

**13.4 Corollary** Denote by  $\text{hor}^{M,F}: \mathfrak{X}(M) \rightarrow \mathfrak{X}_{\text{hor}}(F; \ker T\sigma)$  the horizontal lift of the connection  $\gamma^{M,F} \in \Omega^1(F; \ker \sigma)$ . Then  $\text{hor}^M = \text{hor}^F \circ \text{hor}^{M,F}$ .

**Proof** Fix  $e \in E$  and  $v \in T_{\pi(e)}M$ . By the proof of 13.3, the  $\gamma^{M,F}$ -horizontal lift of  $v$  at  $\rho(e)$  equals  $\text{hor}_{\rho(e)}^{M,F}(v) = T_e\rho \cdot u$  for some  $u \in H_e^M = \ker \gamma^M(e)$ . But then its  $\gamma^F$ -horizontal lift at  $e$  is  $\text{hor}_e^F(\text{hor}_{\rho(e)}^{M,F}(v)) = (\text{hor}_e^F \circ T_e\rho)(u) = u$  since  $u \in H_e^M \subset H_e^F$ , by Theorem 13.1. Therefore  $\langle \gamma^M(e), (\text{hor}^F \circ \text{hor}^{M,F})_e(v) \rangle = \langle \gamma^M(e), u \rangle = 0$ , i.e.,  $(\text{hor}^F \circ \text{hor}^{M,F})_e(v) \in H_e^M$ . From  $\pi = \sigma \circ \rho$ , one gets  $T_e\pi \cdot (\text{hor}^F \circ \text{hor}^{M,F})_e(v) = v$  which proves the corollary. ■

## §14 The Hannay-Berry Connection for General Systems

To start with, let  $\pi : P \times M \rightarrow M$  be a trivial bundle where  $P$  is a symplectic manifold. Endow this bundle with the trivial connection. Let  $H$  be a Hamiltonian on  $P \times M$  defining a completely integrable system for which global action variables exist; thus there is a parametrized torus action on  $P \times M$ . Let  $c$  be a loop at  $m_0$  in  $M$ . The holonomy of  $c$  relative to the Hannay-Berry connection is called the *geometric phase*. We will compare the angular variables in the torus over  $m_0$ , once a complete circuit around the loop  $c$  has been performed. Since the dynamics in the fiber varies as we move in  $c$ , we call the *dynamic phase* the total angular shift due to the frequencies  $\omega^i = \partial H / \partial I^i$  of the integrable system, namely

$$\text{dynamic phase} = \int_0^1 \omega^i(I(c(t)), c(t)) dt . \quad (1)$$

In writing this, we either assume that the loop is contained in a neighborhood whose standard action coordinates are defined, or we postulate the existence of global action variables, which is not always possible due to monodromy (Duistermaat [1980]). In any of these two cases, in completing the circuit  $c$ , we return to the same torus, so a comparison between the angles makes sense. Since the geometric phase is given in such a situation by Hannay's angles as we saw in §10, the total angular shift going once around  $c$  is given by

$$\Delta\theta = \text{dynamic phases} + \text{Hannay's angles} . \quad (2)$$

If we are dealing with a non-integrable system or with a moving system, the fiber dynamics is not so easily accounted for. The general set-up is a Poisson fiber bundle  $\pi : E \rightarrow M$  with a Poisson Ehresmann connection with horizontal lift  $\text{hor}_0$  and a family of Hamiltonian group actions given by a compact Lie group  $G$  defining a parametrized momentum map  $I : E \rightarrow \mathfrak{g}^*$ . We form, for regular values  $\mu \in \mathfrak{g}$ , the parametrized reduced spaces  $I^{-1}(\mu)/G_\mu \rightarrow M$  whose fibers are the reduced spaces (symplectic if the fiber of  $\pi : E \rightarrow M$  were symplectic--Marsden and Weinstein [1974], or Poisson in general--Marsden and Ratiu [1986]). The dynamics on these fibers will, in general, be non-integrable. In the integrable case, these fibers are points and the action of the torus is given by the frequencies  $\omega_1, \dots, \omega_n$  for each frozen value of the parameter. Our goal, as stated at the beginning of the paper, is to determine the geometric (or kinematic) part of the phase shift in the level sets  $I^{-1}(\mu)$ . To do so, consider the tower of bundles

$$\pi_\mu : I^{-1}(\mu) \rightarrow E_\mu := I^{-1}(\mu)/G_\mu , \quad \sigma_\mu : E_\mu \rightarrow M . \quad (3)$$

Assume that  $\langle D_0 I \rangle = 0$ , which guarantees, by the theorems in Sections 6 and 7, the existence and uniqueness of the Hannay-Berry connection on the bundle  $\pi : E \rightarrow M$  which preserves the level sets of  $I$ . Thus we get an induced Hannay-Berry connection with horizontal lift  $\text{hor}$  on the bundle  $\sigma_\mu \circ \pi_\mu = \pi|_{I^{-1}(\mu)} : I^{-1}(\mu) \rightarrow M$ . This, however, is not enough to study the original dynamics on  $I^{-1}(\mu)$  since it ignores the family of reduced dynamics. What we need is a consistent way of lifting curves from the family of reduced spaces  $E_\mu$  to  $I^{-1}(\mu)$ . If there was no parameter involved here, this would be exactly the situation studied in Sections 2 and 3. To put ourselves in this case, we use Sections 2 and 3 to construct on the fiberwise bundles  $\pi_\mu^m : I^{-1}(\mu)^m \rightarrow E_\mu^m$  a connection. We assume this family of connections depends smoothly on  $m \in M$ . In this way, using §2 and §3 we know how to reconstruct the dynamics for each frozen value of  $m$ . Of course, we want to reconstruct it, as  $m$  varies.

The situation described in the previous section deals with such a case: from a connection on a big bundle and fiberwise connections, it gives two connections on the remaining bundles. Theorem 13.1 gives a **generalized Hannay-Berry connection** on the bundle  $\pi_\mu : I^{-1}(\mu) \rightarrow E_\mu$ ; we will denote its horizontal lift by  $\text{hor}^\mu$ . We also get a **base Hannay-Berry connection** on the fiber bundle  $\sigma_\mu : E_\mu \rightarrow M$  by Theorem 13.3; its horizontal lift will be denoted by  $\text{hor}^M$ . If the  $G_\mu$ -actions restricted to each fiber of  $\sigma_\mu \circ \pi_\mu : I^{-1}(\mu) \rightarrow M$  define principal bundles, we conclude that  $\pi_\mu : I^{-1}(\mu) \rightarrow E_\mu$  is a principal  $G_\mu$ -bundle and therefore the generalized Hannay-Berry connection is a principal connection. Consequently, its holonomy can be computed by the method of Proposition 4.1, or if  $G_\mu$  is abelian, one can give an explicit formula in terms of its curvature, as in Corollary 4.2 (see equation (11)).

For connections induced by Cartan connections, the generalized Hannay-Berry connection can be explicitly computed as follows. Let  $E = T^*Q \times M$ ,  $G$  be a Lie group acting on  $Q$ , whose lifted action defines an equivariant momentum map regarded as a parametrized momentum map  $I : T^*Q \times M \rightarrow \mathfrak{g}^*$ , depending trivially on  $M$ . Then we get the bundles

$$\pi_\mu : I^{-1}(\mu) \times M \rightarrow I^{-1}(\mu)/G_\mu \times M \quad \text{and} \quad \sigma_\mu : I^{-1}(\mu)/G_\mu \times M \rightarrow M,$$

the first one being a principal  $G_\mu$ -bundle and the second a trivial bundle. Let  $\gamma$  denote the connection one-form of the induced Hannay-Berry connection given by the Cartan connection  $\gamma_0$  on  $\pi : T^*Q \times M \rightarrow M$  (see Corollary 6.5). We have

$$\begin{aligned} \gamma(\alpha_q, m) \cdot (U_{\alpha_q}, u_m) &= \gamma_0(\alpha_q, m) \cdot (U_{\alpha_q}, u_m) - X_{K \cdot T\pi(U_{\alpha_q}, u_m)}(\alpha_q, m) \\ &= (U_{\alpha_q} + X_{\mathcal{P}_{u_m}}(\alpha_q), 0) - (X_{K \cdot u_m}(\alpha_q), 0) \\ &= (U_{\alpha_q} + X_{\langle \mathcal{P}_{u_m} \rangle}(\alpha_q), 0) \end{aligned} \tag{4}$$

by (4) of §11 and Proposition 11.4 ii.

Now let  $\kappa \in \Omega^1(Q; \mathfrak{g}_\mu)$  be a principal connection on the bundle  $Q \rightarrow Q/G_\mu$  and let  $\kappa^\mu$  be the induced connection on  $I^{-1}(\mu) \rightarrow I^{-1}(\mu)/G_\mu$  (Corollary 2.3). Denote by  $\tau: T^*Q \rightarrow Q$  the cotangent bundle projection and extend  $\kappa^\mu$  trivially to the bundle  $I^{-1}(\mu) \times M \rightarrow I^{-1}(\mu)/G_\mu \times M$ . Think of  $\kappa^\mu$  as a family of connection parametrized (in a trivial way) by  $M$ . Then Theorem 13.1 defines the connection one-form of the generalized Hannay-Berry connection:  $\chi(\alpha_q, m) = \kappa^\mu(\alpha_q, m) \circ \gamma(\alpha_q, m)$ , i.e., by (4)

$$\begin{aligned} \chi(\alpha_q, m) \cdot (U_{\alpha_q}, u_m) &= \kappa^\mu(\alpha_q, m) \cdot (\gamma(\alpha_q, m) \cdot (U_{\alpha_q}, u_m)) \\ &= \kappa^\mu(\alpha_q, m) \cdot (U_{\alpha_q} + X_{\langle \mathcal{P} \cdot u_m \rangle}(\alpha_q), 0) \\ &= \kappa(q) \cdot T_{\alpha_q} \tau(U_{\alpha_q} + X_{\langle \mathcal{P} \cdot u_m \rangle}(\alpha_q)), \end{aligned} \quad (5)$$

where  $\tau: T^*Q \rightarrow Q$  is the projection. We proved the following

**14.1 Theorem** *The generalized HB-connection on  $\pi_\mu: I^{-1}(\mu) \times M \rightarrow I^{-1}(\mu)/G_\mu \times M$  induced by a principal connection  $\kappa \in \Omega^1(Q; \mathfrak{g}_\mu)$  and the Cartan connection on  $T^*Q \times M \rightarrow M$  is a principal connection  $\chi \in \Omega^1(I^{-1}(\mu) \times M; \mathfrak{g}_\mu)$  given by (5).*

In summary, we have connections on all levels of the tower of bundles

$$\begin{array}{c} I^{-1}(\mu) \times M \\ \downarrow \\ I^{-1}(\mu)/G_\mu \times M \\ \downarrow \\ M \end{array}$$

We expect this general context is important for systems like coupled rigid bodies that have joint controls and are subject to overall motions, such as the space telescope. The joint controls can prescribe a motion in  $I^{-1}(\mu)/G_\mu$ , while the overall motion might be the system in orbit about the Earth. The reconstruction of the dynamics on  $I^{-1}(\mu) \times M$  then involves both the cotangent bundle connection on  $I^{-1}(\mu) \rightarrow I^{-1}(\mu)/G_\mu$  (either by Section 2B or 2C) and on  $I^{-1}(\mu) \times M \rightarrow M$  by the Cartan-Hannay-Berry connection.

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